

Tesis Doctoral

**Interactions between order and
algebra in Functional Analysis:
From regular operators
to f -algebras**

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*A mis abuelos:
Adela, Bienvenido,
Marcelino y Rosario*

ABSTRACT

This thesis investigates the structure and representation theory of Banach lattice algebras and Banach f -algebras. Its main motivation is the question: is every Banach lattice algebra isomorphic to a subalgebra and sublattice of the space of regular operators on some order complete Banach lattice?

We are able to settle this question in the negative: there exists a two-dimensional Banach lattice algebra with identity that admits no faithful representation as regular operators on any order complete Banach lattice. The proof builds on top of Wickstead's work, and in particular confirms his conjecture on positive projections with constant diagonal.

To better understand the interplay between the order and algebraic structures in spaces of regular operators, we study their band projections. We introduce the class of inner band projections and show that every band projection on the regular operators is inner if and only if the underlying Banach lattice is atomic and order continuous. We also characterize the multiplication operators that are band projections. Finally, we carry these investigations into the more abstract setting of Banach lattice algebras.

Representations of Banach lattice algebras in spaces of continuous functions are also considered. More precisely, we show that a Banach lattice algebra can be isometrically embedded as a closed sublattice-algebra of a space of continuous functions if and only if it has an approximate order unit that is at the same time an approximate algebraic identity.

For Banach f -algebras, we classify all f -algebra products on AL and AM-spaces, and construct the free normed and Banach f -algebras ($\text{FN}fA[E]$ and $\text{FB}fA[E]$) generated by a Banach space E . Through the development of new techniques for Banach f -algebras, we prove that $\text{FN}fA[E]$ can be represented inside the continuous functions on the unit ball of E^* . We also provide a description, up to isomorphism, of $\text{FB}fA[E]$ for finite-dimensional E . Finally, we characterize the lattice-ordered algebras that admit a functional calculus.

Keywords: regular operator, Banach lattice algebra, f -algebra, Banach f -algebra

RESUMEN

Esta tesis investiga la estructura y la teoría de representación de las álgebras-retículos de Banach y de las f -álgebras de Banach. Su principal motivación es la pregunta: ¿es toda álgebra-retículo de Banach isomorfa a una subálgebra y subretículo del espacio de operadores regulares sobre algún retículo de Banach orden completo?

En este trabajo, seremos capaces de resolver esta pregunta en sentido negativo: existe un álgebra-retículo de Banach bidimensional con identidad que no admite ninguna representación fiel como operadores regulares sobre ningún retículo de Banach orden completo. La demostración se apoya en el trabajo de Wickstead y, en particular, confirma su conjetura sobre las proyecciones positivas con diagonal constante.

Para entender mejor la interacción entre las estructuras de orden y algebraica en los espacios de operadores regulares, estudiamos sus proyecciones en banda. Introducimos la clase de proyecciones en banda internas y demostramos que toda proyección en banda sobre los operadores regulares es interna si y solo si el retículo de Banach subyacente es atómico y orden continuo. Caracterizamos también los operadores de multiplicación que son proyecciones en banda. Finalmente, llevamos estas investigaciones al marco más abstracto de las álgebras-retículos de Banach.

También consideramos las representaciones de las álgebras-retículos de Banach en espacios de funciones continuas. Más concretamente, demostramos que un álgebra-retículo de Banach puede ser embebida isométricamente como subálgebra y subretículo cerrado en un espacio de funciones continuas si y solo si tiene una unidad de orden aproximada que es al mismo tiempo una unidad algebraica aproximada.

Para las f -álgebras de Banach, clasificamos todos los productos de f -álgebra sobre los AL y AM-espacios, y construimos las f -álgebras normada y de Banach libres ($\text{FN}fA[E]$ y $\text{FB}fA[E]$) generadas por un espacio de Banach E . Mediante el desarrollo de nuevas técnicas para f -álgebras de Banach, demostramos que $\text{FN}fA[E]$ puede representarse dentro de las funciones continuas sobre la bola unidad de E^* . Asimismo, proporcionamos una descripción, salvo isomorfismo, de $\text{FB}fA[E]$ para E de dimensión finita. Finalmente, caracterizamos las álgebras reticuladas que admiten un cálculo funcional.

Palabras clave: operador regular, álgebra-retículo de Banach, f -álgebra, Banach f -álgebra

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INTRODUCTION

According to [Kei95], the origins of ordered algebraic structures go back to two independent roots:

- (i) Algebra, in particular arithmetic, divisibility and valuation theory for integral domains. In his 1897 paper [Ded97], Dedekind introduced lattices from an axiomatic point of view (that is, as sets endowed with a meet and a join, satisfying certain axioms) and studied some of their properties, with the goal of understanding lattice-ordered abelian groups.
- (ii) Analysis, in particular the theory of function spaces, positive linear functionals and operators. The seminal talk by F. Riesz at the International Mathematical Congress in Bologna in 1928, where a program for an axiomatic theory of vector lattices was outlined (these ideas were later published in more detail in [Rie40]), ignited a study of these structures that was carried on in the next decade by Freudenthal, Stone, Kantorovich, Kakutani, Nakano, and several others.

By the 1940s, it was clear that a systematic study of lattice-ordered algebraic structures was needed. In 1942, Birkhoff's seminal paper [Bir42] established lattice-ordered groups as an algebraic object in their own right. More than a decade later, in 1956, this was followed by Birkhoff and Pierce's theory of lattice-ordered rings [BP56]. In 1967, in the classical *Lattice Theory* book [Bir61], Birkhoff collected the early developments in the theories of lattices, vector lattices, lattice-ordered groups, rings and algebras (and also f -algebras; see below).

From the analytic point of view, it was soon realized that most of the classical examples of Banach spaces also carried a vector lattice structure. It was thus natural to introduce a (complete) norm into the theory of vector lattices, giving birth to the theory of Banach lattices. An account of the classical Banach lattice theory can be found in the monographs by Lindenstrauss and Tzafriri [LT79], Schaefer [Sch74], and Meyer-Nieberg [MN91].

During the 1970s, 80s, and 90s, a big part of the development of lattice-ordered algebras came from a Dutch school of mathematicians: Huijsmans, de Pagter, van Rooij, Buskes, Luxemburg, and Zaanen, among others. Other important contributors were Henriksen, Isbell, D. G. Johnson, Bernau, Wickstead, and Keimel. It was also during these years that the notion of *Banach lattice algebra*, meaning a lattice-ordered algebra with a complete submultiplicative lattice norm, started to appear. One of the main contributors to this early theory of Banach lattice algebras was Scheffold [Sch81, Sch84, Sch91, Sch93].

However, it was not until the *Ordered Banach Algebras* workshop held in Leiden in 2014, and the problem lists [Wic17a, Wic17b] subsequently published by Wickstead, that the theory of Banach lattice algebras really took off. One of the most prominent questions in both lists is the following.

Main Question. Is every Banach lattice algebra isomorphic to a subalgebra and sublattice of the space of regular operators on some (order complete) Banach lattice?

This pursuit of concrete realization is heavily inspired by foundational representation theorems in operator algebras. It is reminiscent of the Gelfand–Naimark–Segal theorem, which establishes that any abstract C^* -algebra can be faithfully represented as a closed $*$ -subalgebra of bounded operators on a Hilbert space. A positive answer to the representation problem for Banach lattice algebras would provide a similar structural cornerstone, allowing abstract lattice algebras to be analysed through the concrete, well-understood properties of regular operators on order complete Banach lattices.

This thesis aims to contribute to the study of lattice-ordered algebraic structures arising in Functional Analysis. Wickstead’s question lists are our main source of motivation across the different lines of research. More precisely, when starting this thesis our goal was to better understand the Main Question, and to see whether we could make any progress on the differences (if any) between general abstract Banach lattice algebras and sublattice-algebras of spaces of regular operators.

This is the main focus of Chapters 2 and 3, where we approach the question through the study of band projections, a class of operators in which the algebraic and order structures interact particularly nicely. Chapter 2 studies bands and band projections in the algebra $\mathcal{L}_r(X)$ of regular operators on an order complete Banach lattice X . We introduce the *inner band projections*: given a pairwise disjoint family $\{P_\lambda\}_{\lambda \in \Lambda}$ of band projections on X and a subset $\Gamma \subseteq \Lambda \times \Lambda$, the assignment for positive operators $T \mapsto \bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta$ extends to a band projection on $\mathcal{L}_r(X)$ (Theorem 2.1). The inner band projections associated with $\{P_\lambda\}_{\lambda \in \Lambda}$ form a Boolean algebra isomorphic to $2^{\Lambda \times \Lambda}$ (Proposition 2.10). One of the most remarkable results in this chapter is the characterization of the Banach lattices X for which every band projection on $\mathcal{L}_r(X)$ is inner: this happens precisely when X is both atomic and order continuous (Theorem 2.19). We also characterize the multiplication operators on $\mathcal{L}_r(X)$ that are themselves band projections (Theorem 2.35).

In Chapter 3 we transfer this study of band projections to abstract Banach lattice algebras. In a Banach lattice algebra with identity, the natural generalization of band projections is provided by the so-called order idempotents. In a general Banach lattice algebra, we use the previous characterization of multiplication operators to formulate a corresponding notion of band projection. Central to this study is the ideal generated by the identity in a Banach lattice algebra, which is the natural generalization of the center of an order complete Banach lattice. It enjoys similarly good properties: it is lattice-algebra iso-

metric to a space of continuous functions, it is a projection band, and it is an inverse-closed subalgebra (Theorem 3.1). Building on these tools, we generalize the construction of inner band projections to arbitrary order complete Banach lattice algebras (Theorem 3.31).

Chapter 4 plays a special role. It is the chapter in which we have the privilege of solving the Main Question. More precisely, we prove the following.

Main Theorem. *There exists a Banach lattice algebra of dimension 2 with identity that does not admit a faithful representation on any order complete vector lattice.*

We do so by building on Wickstead’s work in [Wic17c], where he reduced the representability of a certain family of Banach lattice algebras of dimension 2 to the following operator-theoretic question.

Wickstead’s Question. Let X be an order complete Banach lattice, and let $P: X \rightarrow X$ be a positive projection for which there exists an $\alpha \geq 0$ satisfying:

- (i) $\alpha \operatorname{id}_X \leq P$, and
- (ii) whenever T is a positive operator such that both $T \leq \lambda \operatorname{id}_X$, for some $\lambda \geq 0$, and $T \leq P$, then also $T \leq \alpha \operatorname{id}_X$.

Which values can α take?

Wickstead conjectured that α must either be 0 or $1/n$, for some $n \in \mathbb{N}$, and proved it for finite-dimensional X . In Corollary 4.15 we show that this is true for general X , and as a consequence we derive the Main Theorem.

Chapter 5 pursues representability from a different angle. The negative answer to the Main Question closes the route to a universal embedding into $\mathcal{L}_r(X)$, but the intrinsic representation-theoretic questions—those concerning modules, submodules, quotients, and the structure of semisimple objects—are largely untouched by it. This chapter is a first systematic step in that direction. Already at the level of the basic definitions it becomes apparent that, when an order structure is added to the classical setting of representation theory, each notion splits into several distinct grades: ordered modules can be regular, positive, or lattice, and submodules can be subspaces, sublattices, ideals, or bands. We develop the corresponding analogues of the foundational results in classical representation theory, including invariant ideals and bands and the associated quotients (Section 5.2), an ordered version of Schur’s lemma (Theorem 5.17), and an ordered version of the Wedderburn–Artin theorem (Theorem 5.27).

Chapter 6 attacks a different representation problem. Instead of trying to represent the elements of a Banach lattice algebra A as regular operators, we try to represent them as continuous functions. More precisely, we ask when A can be lattice-algebra isometrically identified with a closed sublattice-algebra of $C(K)$. We show that this happens exactly when A is an AM-algebra with approximate unit: a Banach lattice algebra possessing a positive net that is at once an approximate order unit (i.e., $x^*(e_\gamma) \rightarrow \|x^*\|$ for every positive functional x^*) and

an approximate algebraic identity (Theorem 6.15). In the case with identity, the statement collapses to the much sharper assertion that every AM-algebra with identity (i.e., every AM-space in which the order unit is an algebraic identity) is lattice-algebra isometric to $C(K)$ itself, with the identity corresponding to $\mathbb{1}_K$. The proof rests on Kakutani's representation theorem, Arens products, and a result of Martignon: the pointwise product is the only product on $C(K)$ with identity $\mathbb{1}_K$ for which, together with pointwise order, $C(K)$ is a lattice-ordered algebra. Furthermore, we characterize $C(K)$ as the unique Banach lattice algebra with identity having this property (Theorem 6.3).

The remaining chapters of the thesis are devoted to a particular class of lattice-ordered algebras called f -algebras. An f -algebra is a lattice-ordered algebra X in which

$$x \wedge y = 0 \quad \text{implies} \quad (zx) \wedge y = 0 = (xz) \wedge y \quad \text{for all } x, y, z \in X_+.$$

The notions of f -algebra and f -ring were introduced by Birkhoff and Pierce back in [BP56]. Since then, these structures have been extensively studied from an algebraic point of view. A comprehensive account of this line of research can be found in the survey [Hen97] and its more than 150 references.

But why were these notions introduced in the first place? To answer this question, let us consider the framework of vector lattices consisting of real-valued functions defined on a set. Most such spaces naturally carry a multiplication operation (typically, the pointwise product) under which the positive cone remains closed. Lattice-ordered algebras provide a first abstraction of this situation. However, Birkhoff and Pierce soon realized that the general theory of lattice-ordered algebras was not particularly tractable, and that a further condition was needed to properly abstract the behavior of function spaces. This missing ingredient was the f -algebra condition: if $x \wedge y = 0$ and $z \geq 0$, then

$$(zx) \wedge y = 0 = (xz) \wedge y.$$

This property encapsulates the idea that the pointwise product of functions preserves their supports. The mathematical fact that led Birkhoff and Pierce to this particular condition is the following: a lattice-ordered algebra is a sub-direct product of totally ordered algebras if and only if it satisfies the f -algebra condition. As a consequence, one of the most celebrated results in the early theory of f -algebras was obtained: every Archimedean f -algebra is associative and commutative (even if associativity is not assumed as part of the definition).

The first application of f -algebras to Operator Theory is due to Bigard and Keimel [BK69] and to Conrad and Diem [CD71], independently: the set of orthomorphisms (order-bounded band-preserving operators) on an arbitrary vector lattice, equipped with the usual order and with composition as product, forms an f -algebra. This observation gave rise to an intensive development of the theory of f -algebras from an operator-theoretic perspective. The works of Huijsmans [Hui85, Hui89], de Pagter [DP81, HdP82, HdP84b, dP84], Bernau [BH90b, BH90a, BH95], van Rooij [BdPvR91], and Wickstead [Wic80, Wic87],

among others, greatly contributed to this progress. More recent developments and applications include: the study of averaging and Reynolds operators [BBT07], the construction of vector lattice powers [BB06], tensor products [BW17, Jab20], and free objects [Kei95, dJ21].

Chapter 7 is also prompted by a question of Wickstead. In [Wic17a, Section 4], Wickstead asked for a description of the most general Banach lattice algebra structure on AM-spaces and AL-spaces. In Chapter 7 we describe the f -algebra products on these spaces. For every AM-space X we construct an associated AM-space W_X and show that the f -algebra products on X correspond bijectively with $(W_X)_+$ (Theorem 7.4); this generalizes the classical parametrization of the f -algebra products on $C(K)$ by elements of $C(K)_+$. We then study the two extremes of this correspondence. When W_X is maximal, X admits a (necessarily unique) product turning it into an AM-algebra; the AM-spaces with this property admit several equivalent descriptions, including being lattice isometric to a closed ideal of some $C(K)$ (Theorem 7.8). When W_X is minimal, X admits only the zero product as an f -algebra product. For AL-spaces, the classification is particularly clean: the f -algebra products on an AL-space are parametrized by the bounded weights on a maximal family of norm-one atoms (Theorem 7.17), and as a consequence an AL-space admits only the zero product if and only if it has no atoms (Corollary 7.18).

Chapter 8 is devoted to the study of free objects in the category of Banach f -algebras. Banach f -algebras were first investigated in the 1980s by Martignon [Mar80] and Scheffold [Sch81, Sch91, Sch93]. It is remarkable how little attention these objects have received, given that Banach lattices have been extensively investigated and that Banach f -algebras are particularly well behaved. First of all, one should note that every Banach lattice equipped with the identically zero multiplication becomes a Banach f -algebra in a standard, if trivial, way. Moreover, every Banach f -algebra with identity is lattice-algebra isomorphic to a space of continuous functions—a fact that directly connects with the original motivation of modeling spaces of functions.

It is natural to assume that, in order to understand a given category, one must understand its free objects. Free groups, algebras, modules, lattices, and Boolean algebras, among others, play a central role throughout algebra and order theory. In topology, the Stone–Čech compactification and the completion of a metric or uniform space can also be viewed as instances of free constructions. A modern and increasingly powerful approach in Functional Analysis is to study free objects in categories related to Banach spaces. Lipschitz free Banach spaces [GK03], holomorphic free Banach spaces [Muj91], and free Banach lattices [ART18] are notable examples (see the survey [GSdHT24] for a general overview). It is therefore natural to attempt to extend the theory of Banach f -algebras in this direction.

The main inspiration for developing free Banach f -algebras comes from the theory of free Banach lattices. The free Banach lattice generated by a set was first introduced in [dPW15]. Although this foundational work established several key properties, many questions remained open, since the authors lacked an explicit description of the norm of such objects. This gap was later filled

in [ART18], where both an explicit construction and a norm were provided for the more general case of the free Banach lattice generated by a Banach space. This breakthrough resolved several of the open problems from [dPW15] and also answered a question posed by J. Diestel concerning weakly compactly generated Banach lattices. Following these two seminal papers, the theory of free Banach lattices has flourished, with significant developments in various directions—for instance, free complex Banach lattices [dHT23], free Banach lattices generated by a lattice [ARA19, AMCRAZ22], and free Banach lattices satisfying convexity conditions [JSLT⁺22]—and with applications to the study of norm-attaining lattice homomorphisms [BGSdH⁺26, DMCRARZ22], the relation between linear and lattice embeddings [AMCRZT22], the construction of push-outs [AT23], and the description of complemented subspaces of Banach lattices [dHT25]. The study of free Banach lattices themselves has also become a subject of great interest [OTTT24, ATV19, GST24, APRA18, DMCRARZ21], as they serve as the canonical tool for understanding the interaction between Banach spaces and Banach lattices.

The study of free Banach f -algebras represents a further step in this direction. Although these objects (and more generally, free Banach lattice algebras) have been considered before [dJ21, Wic17a], no explicit construction is yet known of a free object combining norm, lattice, and algebraic structures. The main obstacle lies in the presence of a product, which renders most known techniques ineffective. Indeed, the existing methods for free Banach lattices often rely on the fact that equations in a vector lattice are positively homogeneous—a property that no longer holds in f -algebras. New approaches are therefore required.

This motivates the research on free Banach f -algebras that constitutes Chapter 8. The chapter develops the construction of the free Banach f -algebra $\text{FB}fA[E]$ generated by a Banach space E in three stages—Archimedean, normed, and Banach—and proves the central result that the free normed f -algebra can be represented in $C(B_{E^*})$ (Corollary 8.51), whereas $\text{FB}fA[E]$ admits an injective lattice-algebra representation in $C(B_{E^*})$ if and only if it is semiprime (Theorem 8.75). Both representability and semiprimeness can be verified explicitly when E is finite-dimensional and when $E = L_1(\mu)$ for an arbitrary measure μ ; in the finite-dimensional case we even obtain a complete description of $\text{FB}fA[E]$ as the space $C([0, 1] \times S_{E^*})$ equipped with pointwise order and a weighted product (Theorem 8.56). Whether $\text{FB}fA[E]$ is semiprime for an arbitrary Banach space E remains open: even though the free normed f -algebra is indeed semiprime, the completion of a semiprime normed f -algebra need not be semiprime (Example 8.79). Along the way, the chapter develops a structure theorem for semiprime Banach f -algebras that is of independent interest, including a remarkable extension result for real-valued lattice-algebra homomorphisms from a closed sublattice-algebra to the whole space (Theorem 8.34).

Chapter 9 is devoted to the study of the lattice-ordered algebras that admit a functional calculus. In [LT20], Laustsen and Troitsky characterized the Archimedean vector lattices that admit a positively homogeneous continuous function calculus. The natural extension of this calculus to the algebraic set-

ting is the polynomial growth continuous function calculus. We characterize the Archimedean lattice-ordered algebras with identity that admit such a calculus (Theorem 9.10 and Corollary 9.17) by an analogue of the condition of Laustsen and Troitsky. Along the way, we identify the free uniformly complete Archimedean f -algebra on finitely many generators (Theorem 9.8), and prove that any vector space admitting a sufficiently strong polynomial growth continuous function calculus is automatically a commutative f -algebra (Theorem 9.21).

The contents of this thesis are based on the following publications and preprints:

- David Muñoz Lahoz and Pedro Tradacete, *Band projections in spaces of regular operators*, Trans. Amer. Math. Soc. **377** (2024), no. 7, 5197–5218. MR [4778072](#)
- David Muñoz Lahoz, *Band projections and order idempotents in Banach lattice algebras*, J. Math. Anal. Appl. **543** (2025), no. 2, part 3, Paper No. 129045, 27. MR [4824648](#)
- David Muñoz Lahoz and Pedro Tradacete, *Banach lattice AM-algebras*, Proc. Amer. Math. Soc. **153** (2025), no. 6, 2565–2577. MR [4892628](#)
- David Muñoz-Lahoz, *f -algebra products on AL and AM-spaces*, Preprint, arXiv:2507.08435. To appear in Math. Z., 2026
- David Muñoz-Lahoz and Pedro Tradacete, *Free banach f -algebras*, Preprint, arXiv:2511.13299, 2026
- David Muñoz-Lahoz, *Wickstead’s conjecture on positive projections and non-representable banach lattice algebras*, Preprint, arXiv:2604.14697, 2026
- David Muñoz-Lahoz, *Lattice-ordered algebras admitting a polynomial growth continuous function calculus*, Preprint, arXiv:2604.20294, 2026
- Marcel de Jeu and David Muñoz-Lahoz, *Representation theory of vector lattice algebras*, Manuscript in preparation

Every chapter, except for the preliminaries, is based on one of the above papers. At the beginning of each chapter, the appropriate reference is provided.

1 PRELIMINARIES

This chapter provides some background on the topics of the thesis: vector and Banach lattices (Section 1.1), positive operator theory (Section 1.2), Banach lattice algebras (Section 1.3), f -algebras (Section 1.4), Banach f -algebras (Section 1.5), and free Banach lattices (Section 1.6). Classical results are often presented inline and without proof. They can all be found in the monographs cited at the beginning of each section.

1.1 VECTOR AND BANACH LATTICES

This section introduces basic terminology and results from the theory of vector and Banach lattices. All the contents can be found in the monographs [AB06, LT79, MN91, Sch74].

1.1.1 The basics

A *vector lattice* (or *Riesz space*) is a real vector space X partially ordered by a relation $\leq$ that is compatible with the linear structure ($x \leq y$ implies $x + z \leq y + z$ and $\lambda x \leq \lambda y$ for every $z \in X$ and every $\lambda \in \mathbb{R}_+$) and under which every two elements $x, y \in X$ admit a supremum $x \vee y$ and an infimum $x \wedge y$. The *positive cone* of X is $X_+ = \{x \in X : x \geq 0\}$. For each $x \in X$ we set

$$x_+ = x \vee 0, \quad x_- = (-x) \vee 0, \quad |x| = x \vee (-x);$$

these are called the *positive part*, *negative part*, and *modulus* of x . Two elements $x, y \in X$ are *disjoint*, written $x \perp y$, if $|x| \wedge |y| = 0$. An *order interval* of X is a set of the form $[a, b] = \{x \in X : a \leq x \leq b\}$ ($a \leq b$ in X); a subset of X is *order bounded* if it is contained in some order interval.

A vector lattice X is *Archimedean* if, for $x, y \in X_+$, the condition $nx \leq y$ for every $n \in \mathbb{N}$ forces $x = 0$. Most, but not all, of the vector lattices that appear in this thesis are Archimedean.

A *normed lattice* is a vector lattice X equipped with a norm $\|\cdot\|$ that is monotone with respect to the modulus, i.e., $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Every normed lattice is Archimedean. If, furthermore, the norm is complete, then X is said to be a *Banach lattice*. Classical examples of Banach lattices, all equipped with the pointwise (resp. almost everywhere) order, are: the space c_0 of real sequences converging to 0 with supremum norm; for $1 \leq p < \infty$ and any set I , the space $\ell_p(I)$ of p -summable functions $I \rightarrow \mathbb{R}$, and the space $\ell_\infty(I)$ of bounded functions $I \rightarrow \mathbb{R}$; for $1 \leq p \leq \infty$ and any measure space (Ω, Σ, μ) , the standard Lebesgue space $L_p(\mu)$; and, for any compact Hausdorff space K , the

space $C(K)$ of continuous real-valued functions on K . In Section 1.6 we will have the opportunity to explore more “exotic” examples.

A linear operator $T: X \rightarrow Y$ between vector lattices is called *positive* if $T(X_+) \subseteq Y_+$. It is a *lattice homomorphism* if $T(x \vee y) = Tx \vee Ty$ for every $x, y \in X$; equivalently, if $T|x| = |Tx|$ for every $x \in X$. Every lattice homomorphism is positive. Between Banach lattices, every positive linear operator is automatically bounded.

Let X be a vector lattice. A vector subspace $Y \subseteq X$ is a *sublattice* if $x \in Y$ implies $|x| \in Y$, an *ideal* if it is also *solid* (that is, $|x| \leq |y|$ and $y \in Y$ imply $x \in Y$), and a *band* if it is an ideal and $\sup_\alpha y_\alpha \in Y$ whenever $(y_\alpha) \subseteq Y$ is an upward directed net whose supremum exists in X . The *disjoint complement* of a subset $S \subseteq X$ is

$$S^d = \{ x \in X : |x| \wedge |s| = 0 \text{ for all } s \in S \};$$

the disjoint complement of any subset is a band.

A band $B \subseteq X$ is a *projection band* if $X = B \oplus B^d$. In this case the linear projection $P: X \rightarrow X$ with range B and kernel B^d is called the *band projection* onto B . On a Banach lattice X , every band projection is automatically positive and contractive, and a band $B \subseteq X$ is a projection band if and only if $\sup\{ y \in B : 0 \leq y \leq x \}$ exists in X for every $x \in X_+$, in which case this supremum is $P(x)$. We denote by $\mathfrak{B}(X)$ the set of projection bands of X and by $\mathfrak{P}(X)$ the set of band projections; both carry a canonical Boolean algebra structure that we recall in more detail in Section 1.2.2.

If J is a closed ideal of a Banach lattice X , then the quotient X/J inherits a Banach lattice structure with positive cone the image of X_+ , and the canonical projection $X \rightarrow X/J$ becomes a contractive lattice homomorphism.

For $x \in X_+$, the *principal ideal* generated by x (i.e., the smallest ideal containing x) is $I_x = \bigcup_{n \in \mathbb{N}} [-nx, nx]$, and the *principal band* generated by x (i.e., the smallest band containing x) is denoted B_x . The principal ideal I_x carries the norm

$$\|y\|_x = \inf\{ \lambda > 0 : |y| \leq \lambda x \}, \quad y \in I_x,$$

which is a lattice norm whenever X is Archimedean; the pair $(I_x, \|\cdot\|_x)$ is then an AM-space with unit x (see Section 1.1.3).

A positive element $e \in X_+$ is called a *strong (order) unit* of X if $X = I_e$, a *weak order unit* if $|x| \wedge e = 0$ implies $x = 0$, and, when X is a Banach lattice, a *quasi-interior point* of X if $\overline{I_e} = X$. A vector sublattice Y of a vector lattice X is *order dense* in X if $Y^d = \{0\}$, equivalently, if for every $0 < x \in X$ there exists $y \in Y_+$ with $0 < y \leq x$.

A non-zero element $a \in X_+$ is an *atom* of X if every $x \in X$ with $0 \leq x \leq a$ is a scalar multiple of a , equivalently, if the principal ideal I_a is one-dimensional. A Banach lattice X is *atomic* (or *discrete*) if the band generated by the atoms is X , and *non-atomic* if it has no atoms. The spaces $\ell_p(I)$ ($1 \leq p \leq \infty$) are atomic, while $L_p(\mu)$ is non-atomic whenever the measure μ is atomless.

Every Archimedean vector lattice X satisfies, for all $x, y, z \in X$,

- (i) $(x \vee y) + z = (x + z) \vee (y + z)$ and $(x \wedge y) + z = (x + z) \wedge (y + z)$;
- (ii) $(x \vee y) + (x \wedge y) = x + y$, $|x| = x_+ + x_-$, and $x = x_+ - x_-$;
- (iii) $x \vee y = \frac{1}{2}(x + y + |x - y|)$ and $x \wedge y = \frac{1}{2}(x + y - |x - y|)$;
- (iv) $|x + y| \leq |x| + |y|$ and $||x| - |y|| \leq |x - y|$.

More generally, every identity that involves finitely many variables, the linear operations (i.e., addition and scalar multiplication), and the lattice operations (i.e., supremum and infimum), and that holds in the reals, must hold in every Archimedean vector lattice. This is a consequence of Kakutani's theorem (see Section 1.1.3).

1.1.2 Order completeness, order continuity, and uniform convergence

A vector lattice X is *order complete* if every subset of X that is bounded above admits a supremum in X , and *σ -order complete* if the same holds for countable subsets bounded above. In the literature, these properties are also referred to as *Dedekind completeness* and *σ -Dedekind completeness*; we stick to “order complete” throughout the thesis.

For example, $L_p(\mu)$ is order complete for every $1 \leq p \leq \infty$; c_0 is order complete; $C(K)$ is order complete if and only if K is extremally disconnected, and σ -order complete if and only if K is basically disconnected. In an order complete Banach lattice every band is a projection band, and the Boolean algebras $\mathfrak{B}(X)$ and $\mathfrak{P}(X)$ are canonically isomorphic through the map that sends each band to its corresponding projection band.

A Banach lattice X has *order continuous norm*, or is *order continuous*, if, for every decreasing net $(x_\alpha) \subseteq X_+$ with $\inf_\alpha x_\alpha = 0$, $\|x_\alpha\| \rightarrow 0$. The spaces $L_p(\mu)$ ($1 \leq p < \infty$) and c_0 are order continuous, while $L_\infty(\mu)$, ℓ_∞ , and $C(K)$ for K infinite are not. Every order continuous Banach lattice is order complete.

We close this subsection with the notions of uniform convergence and uniform completeness. Let X be an Archimedean vector lattice and $e \in X_+$. A net $(x_\alpha) \subseteq X$ *converges uniformly to x with regulator e* , written $x_\alpha \xrightarrow{u} x (e)$, if for every $\varepsilon > 0$ there exists an index α_0 such that $|x_\alpha - x| \leq \varepsilon e$ for every $\alpha \geq \alpha_0$. Uniform convergence with regulator e is exactly norm convergence in the principal ideal $(I_e, \|\cdot\|_e)$; in particular, uniform convergence is sequential in nature. The vector lattice X is *uniformly complete* if $(I_e, \|\cdot\|_e)$ is complete for every $e \in X_+$. Every Banach lattice is uniformly complete. Throughout the thesis, when we speak of a uniformly complete vector lattice, we always assume it to be Archimedean.

1.1.3 AL and AM-spaces

Two distinguished classes of normed lattices admit an explicit representation as classical function spaces.

Definition 1.1. A normed lattice X is an

- (i) *AM-space* if $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in X_+$;
- (ii) *AL-space* if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$.

An AM-space X has *unit* $e \in X_+$ if e is a strong order unit and $\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda e\}$ for every $x \in X$; equivalently, if the closed unit ball of X coincides with the order interval $[-e, e]$.

The prototypical AM-space with unit is $C(K)$, with unit the constant one function $\mathbb{1}_K$; more generally, $L_\infty(\mu)$ on a measure space (Ω, Σ, μ) and the bounded continuous functions $C_b(\Omega)$ on a topological space Ω are examples of AM-spaces with unit. The space c_0 is an AM-space without unit. The prototypical AL-space is $L_1(\mu)$. Another example is the space $M(K)$ of finite signed regular Borel measures on a compact Hausdorff space K .

The following theorem summarizes the Kakutani representation theorems for AL and AM-spaces.

Theorem 1.2 (Kakutani [[Kak41b](#), [Kak41a](#)]).

- (i) *Every AM-space with unit is lattice isometric to $C(K)$ for some compact Hausdorff space K , with the unit corresponding to the constant one function $\mathbb{1}_K$.*
- (ii) *Every AM-space is lattice isometric to a closed sublattice of $C(K)$ for some compact Hausdorff space K . More precisely, there exist a family $\{(t_i, s_i)\}_{i \in I} \subseteq K \times K$ and scalars $\{\lambda_i\}_{i \in I} \subseteq [0, 1]$ such that the AM-space identifies, lattice isometrically, with the closed sublattice*

$$\{f \in C(K) : f(t_i) = \lambda_i f(s_i) \text{ for all } i \in I\}.$$

- (iii) *Every AL-space is lattice isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ) .*

The representation theorem for AM-spaces with unit can be used to provide “local” representations of Archimedean vector lattices as spaces of continuous functions. More precisely, let X be an Archimedean vector lattice, and let $u \in X_+$. Then the principal ideal $(I_u, \|\cdot\|_u)$ is a normed lattice with strong unit u that can be completed to an AM-space with unit u . By Kakutani’s representation theorem, there exists a compact Hausdorff space K_u and a lattice isometric embedding $I_u \rightarrow C(K_u)$ with dense range. When X is a uniformly complete Archimedean vector lattice (in particular, when it is a Banach lattice), I_u is lattice isometric to $C(K_u)$.

This local representation can be used to prove the principle, anticipated in Section 1.1.1, that every lattice identity that holds in $\mathbb{R}$ must also hold in every Archimedean vector lattice. Given finitely many elements $x_1, \dots, x_n$ of an Archimedean vector lattice X , taking $u = |x_1| + \dots + |x_n|$ all x_i lie in I_u and may be regarded as continuous functions on a certain compact Hausdorff space K_u . As lattice and linear operations correspond pointwise on $C(K_u)$, every identity

built from $+$, $\vee$, $\wedge$ and real scalars that holds for real numbers automatically holds for $x_1, \dots, x_n$.

Another important fact is that AL and AM-spaces are dual to each other, in the following sense.

Proposition 1.3. *Let X be a Banach lattice.*

- (i) *X is an AL-space if and only if X^* is an AM-space with unit.*
- (ii) *X is an AM-space if and only if X^* is an AL-space.*

1.1.4 Complexification

Let X be a Banach lattice. The *complexification* of X , as a vector space, is the additive group $X \times X$ with scalar multiplication

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x),$$

where $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. The group $X \times X$ with this scalar multiplication is a complex vector space, usually denoted by $X_{\mathbb{C}}$. Identifying $x \in X$ with $(x, 0) \in X \times X$ and ix with $(0, x)$, we write $x + iy$ instead of $(x, y) \in X \times X$.

Let X be a Banach lattice. For $z = x + iy \in X_{\mathbb{C}}$ define

$$|z| = \sup\{x \cos \phi + y \sin \phi : 0 \leq \phi \leq 2\pi\}.$$

This supremum exists in X , and it satisfies:

- (i) $|z| = 0$ if and only if $z = 0$,
- (ii) $|\lambda z| = |\lambda| |z|$ for all $\lambda \in \mathbb{C}$ and $z \in X_{\mathbb{C}}$,
- (iii) $|z + w| \leq |z| + |w|$ for all $z, w \in X_{\mathbb{C}}$.

Moreover, the map $z \mapsto |||z|||$ defines a norm on $X_{\mathbb{C}}$, and $X_{\mathbb{C}}$ with this norm is a complex Banach space. A *complex Banach lattice* is a Banach space of the form $X_{\mathbb{C}}$, for a certain Banach lattice X , with the norm and modulus introduced in the previous proposition.

1.2 SOME POSITIVE OPERATOR THEORY

This section reviews some of the basic material on operator theory in vector and Banach lattices that will motivate our study of Banach lattice algebras and f -algebras later on. The material of this section can be found in [AA02a].

1.2.1 Regular operators

Let X and Y be Banach lattices, and let $\mathcal{L}(X, Y)$ denote the set of (linear and bounded) operators from X to Y . We say that an operator $T: X \rightarrow Y$ is *positive* if $T(X_+) \subseteq Y_+$. The set of positive operators $\mathcal{L}_+(X, Y)$ forms a cone in $\mathcal{L}(X, Y)$.

With this cone, $\mathcal{L}(X, Y)$ is an ordered Banach space, but it need not be a lattice; in fact, it may even be the case that $\mathcal{L}_+(X, Y) - \mathcal{L}_+(X, Y)$ is a proper subset of $\mathcal{L}(X, Y)$. The elements of $\mathcal{L}_+(X, Y) - \mathcal{L}_+(X, Y)$ are called *regular operators*, and the set of regular operators is denoted by $\mathcal{L}_r(X, Y)$. Regular operators are those operators that can be written as the difference of two positive operators; equivalently, an operator T is regular if there exists a positive operator $S \in \mathcal{L}_+(X, Y)$ such that $T \leq S$.

The space of regular operators $\mathcal{L}_r(X, Y)$ need not be a Banach lattice either. However, when Y is order complete, $\mathcal{L}_r(X, Y)$ is always a Banach lattice. More precisely:

Theorem 1.4 (Riesz–Kantorovich). *Let X and Y be Banach lattices. If Y is order complete, then $\mathcal{L}_r(X, Y)$ is a Banach lattice, with supremum and infimum given by*

$$(S \vee T)x = \sup\{Su + Tv : u, v \in X_+, u + v = x\}, \quad (1.1)$$

$$(S \wedge T)x = \inf\{Su + Tv : u, v \in X_+, u + v = x\}, \quad (1.2)$$

where $S, T \in \mathcal{L}_r(X, Y)$ and $x \in X_+$, and with norm

$$\|T\|_r = \| |T| \|. \quad (1.3)$$

Moreover, $\mathcal{L}_r(X, Y)$ is order complete.

Equations (1.1) and (1.2) and their derivatives (e.g., $|T| = \sup\{|Tu| : -x \leq u \leq x\}$) are called the *Riesz–Kantorovich formulae*. The norm defined in (1.3) is called the *regular norm*.

A particularly important application of the Riesz–Kantorovich formulae is to the dual $X^* = \mathcal{L}(X, \mathbb{R})$ of a Banach lattice X . With pointwise order ($x^* \leq y^*$ if and only if $x^*(x) \leq y^*(x)$ for all $x \in X_+$), the dual is itself an order complete Banach lattice, with lattice operations given by

$$\begin{aligned} (x^* \vee y^*)(x) &= \sup\{x^*(u) + y^*(v) : u, v \in X_+, u + v = x\}, \\ (x^* \wedge y^*)(x) &= \inf\{x^*(u) + y^*(v) : u, v \in X_+, u + v = x\}, \end{aligned}$$

for $x \in X_+$. Iterating, the *bidual* X^{**} is again an order complete Banach lattice, and the canonical isometric embedding $J: X \rightarrow X^{**}$ is a lattice homomorphism; we identify X with the sublattice $J(X)$ whenever convenient. The second adjoint T^{**} of a lattice homomorphism between Banach lattices is again a lattice homomorphism.

When $X = Y$, we will simplify our notation from $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$, from $\mathcal{L}_r(X, X)$ to $\mathcal{L}_r(X)$, and so on. In this case, the space $\mathcal{L}(X)$ is a Banach algebra with multiplication given by composition, and norm given by operator norm. It is clear that the composition of positive operators is again positive. From this, and the fact that the modulus is submultiplicative (i.e., $|TS| \leq |T||S|$), it follows that $\mathcal{L}_r(X)$ is a Banach algebra with respect to the regular norm. If X is order complete, we have seen that $\mathcal{L}_r(X)$ is a Banach lattice. The simultaneous existence of these two structures will be further discussed in Section 1.3.

1.2.2 Band projections

The interaction between the lattice and algebraic structures of $\mathcal{L}_r(X)$ manifests itself clearly when we study band projections. It turns out that band projections can be nicely characterized in terms of the algebraic and order structure in $\mathcal{L}(X)$.

Theorem 1.5. *Let X be a Banach lattice. An operator $P: X \rightarrow X$ is a band projection if and only if $P^2 = P$ and $0 \leq P \leq \text{id}_X$.*

Finite infima and suprema of band projections yield new band projections. Surprisingly enough, lattice operations between band projections are determined by the algebraic and linear operations only.

Theorem 1.6. *Let X be a Banach lattice. The set of band projections on X forms a Boolean algebra with operations*

$$\overline{P} = \text{id}_X - P, \quad P_1 \vee P_2 = P_1 + P_2 - P_1 P_2, \quad P_1 \wedge P_2 = P_1 P_2,$$

minimum the zero map, and maximum the identity map. Moreover, suprema and infima coincide with those in $\mathcal{L}_r(X)$.

A direct consequence of the previous theorem is that band projections commute. Note here the interplay between the lattice and algebraic operations: we are able to express the lattice operations in purely algebraic terms, and from this we are able to deduce an algebraic property.

1.2.3 The center of a Banach lattice

Central operators and the center of a Banach lattice have been extensively studied in the literature (see [AA02a, Section 3.3]). Here we only recall some basic definitions and a couple of results that will be useful later.

Definition 1.7. Let X be a Banach lattice. An operator $T: X \rightarrow X$ is called *central* if there exists some scalar $\lambda > 0$ such that $|Tx| \leq \lambda|x|$ for all $x \in X$. The collection of all central operators is denoted $\mathcal{Z}(X)$ and is called the *center* of X .

For example, central operators on $C(K)$ are precisely the operators of the form $M_f(g) = fg$, where $f, g \in C(K)$. Note also that band projections are positive central operators. When X is order complete, $\mathcal{Z}(X)$ is just the ideal generated by the identity in the Banach lattice $\mathcal{L}_r(X)$.

Central operators have nice properties: they are regular and order continuous, and positive central operators are also lattice homomorphisms. The following theorem collects some of the most important properties of the center.

Theorem 1.8 (Wickstead [Wic77]). *Let X be a Banach lattice.*

- (i) The center $\mathcal{Z}(X)$, equipped with the operator norm, is an AM-space with unit id_X . In particular,

$$\|T\| = \| |T| \| = \inf\{ \lambda > 0 : |T| \leq \lambda \text{id}_X \}$$

for every $T \in \mathcal{Z}(X)$, and $\mathcal{Z}(X)$ is lattice-algebra isometric to $C(K)$ for a certain compact Hausdorff space K .

- (ii) The center $\mathcal{Z}(X)$ is an inverse closed subalgebra of $\mathcal{L}_r(X)$ (i.e., if a central operator is invertible, its inverse is also central).
- (iii) When X is order complete, the center $\mathcal{Z}(X)$ is the band generated by id_X in $\mathcal{L}_r(X)$. In particular, it is a projection band.

When X is order complete, the band projection onto the center is usually denoted by $\mathcal{D}$ and is called the *diagonal projection* or the *projection onto the diagonal*. The reason for this terminology is that, when X has finite dimension n , $\mathcal{L}_r(X)$ is the space of n by n real matrices, and $\mathcal{D}$ is the map that keeps the diagonal of a matrix and sets all the non-diagonal entries equal to 0.

An operator $T: X \rightarrow X$ on a vector lattice X is said to be

- (i) an *orthomorphism* if $|x| \wedge |y| = 0$ implies $|x| \wedge |Ty| = 0$ for every $x, y \in X$;
- (ii) *band preserving* if $T(B) \subseteq B$ for every band B in X .

It is immediate that central operators are orthomorphisms, and that orthomorphisms are band preserving. Not trivial, but true, is that for Banach lattices these three classes of operators coincide.

Theorem 1.9 (Abramovich–Veksler–Koldunov [AVK79]). *For an operator $T: X \rightarrow X$ on a Banach lattice X the following are equivalent.*

- (i) T is a central operator.
- (ii) T is an orthomorphism.
- (iii) T is band preserving.

Orthomorphisms will play a central role in our exposition. We will have the opportunity to study them in more detail in Section 1.4.2, once f -algebras have been introduced.

1.2.4 Complexification and spectra

We now introduce regular operators between complex Banach lattices $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$. For every (complex) linear operator $T: X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ there exist unique (real) linear operators $T_1, T_2: X \rightarrow Y$ such that $Tx = T_1x + iT_2x$ for all $x \in X$. The operator T_1 is called the *real part* of T , while the operator T_2 is called the *imaginary part* of T . Thus $\mathcal{L}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is isomorphic to the complexification of $\mathcal{L}(X, Y)$. An operator $T: X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ is said to be *real valued* if $T(X) \subseteq Y$. If

T is real valued and $T(X_+) \subseteq Y_+$, then T is said to be *positive*. An operator $T: X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ is said to be *regular* if both its real and imaginary parts are regular. When Y is order complete, the space of regular operators between $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ may be identified with the complexification of the Banach lattice $\mathcal{L}_r(X, Y)$.

Given a regular operator $T: X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$, denote by $\sigma(T)$ its spectrum. The following result characterizes the lattice isomorphisms contained in the center in terms of their spectrum, and will be needed later in the discussion.

Theorem 1.10 (Schaefer–Wolff–Arendt [SWA78]). *Let X be a Banach lattice, and let T be a lattice isomorphism of $X_{\mathbb{C}}$. Then $T \in \mathcal{Z}(X_{\mathbb{C}})$ if and only if $\sigma(T) \subseteq \mathbb{R}_+$.*

1.3 BANACH LATTICE ALGEBRAS

In the space of regular operators on an order complete Banach lattice, a Banach lattice structure (given by the cone of positive operators) and a Banach algebra structure (given by composition) coexist. We want to study more general spaces in which these two structures coexist. This will elucidate which properties of the space of regular operators do not really depend on the fact that they are regular operators on a Banach lattice, but rather on the algebraic and lattice structures of the space.

But first, we need to define Banach lattice algebras. It seems like there is no consensus in the literature on the notion of Banach lattice algebra with identity, so we follow the one suggested by A. W. Wickstead in [Wic17a].

Definition 1.11. A *Banach lattice algebra* is a Banach lattice A together with a product $\ast: A \times A \rightarrow A$ such that

- (i) $(A, \ast)$ is a Banach algebra,
- (ii) if $x, y \in A_+$, then $x \ast y \in A_+$.

When the product $\ast$ has an identity e of norm one, we say that A is a *Banach lattice algebra with identity e* . When the product $\ast$ is commutative, we say that A is a *commutative Banach lattice algebra*.

It is important to note the following consequence of the definition.

Proposition 1.12 ([BH90b, Theorem 2]). *The identity in a Banach lattice algebra is positive.*

Let us pause here to present some classical examples of Banach lattice algebras. We shall resort to them when presenting examples and counterexamples in our exposition.

Example 1.13. (i) Let K be a compact Hausdorff space. The space $C(K)$ of real-valued continuous functions on K is a commutative Banach lattice algebra with pointwise order and product, and supremum norm. The identity is the constant one function.

- (ii) Let X be an order complete Banach lattice. We have seen in Section 1.2.1 that the space of regular operators $\mathcal{L}_r(X)$ is a Banach lattice, and at the same time a Banach algebra, in which the composition of positive operators is positive. It is, therefore, a Banach lattice algebra, with identity id_X . This Banach lattice algebra need not be commutative.
- (iii) Let G be a locally compact group, let λ be a left invariant Haar measure, and consider the Banach lattice $L_1(G) = L_1(\lambda)$. The convolution multiplication on $L_1(G)$, defined for $\phi, \psi \in L_1(G)$ by

$$(\phi * \psi)(g) = \int_G \phi(gh)\psi(h^{-1}) d\lambda(h), \quad \text{where } g \in G,$$

makes $L_1(G)$ into a Banach algebra. It is easy to check that the product of positive elements is positive, so $L_1(G)$ is a Banach lattice algebra. This Banach lattice algebra is commutative if and only if G is abelian, and has an identity if and only if G is discrete.

- (iv) Let $M(G)$ be the space of signed regular Borel measures on a locally compact group G . Then $M(G)$ is a Banach lattice if we set $\mu \geq \nu$, $\mu, \nu \in M(G)$, whenever $\mu(A) \geq \nu(A)$ for every Borel set A , and give it the norm $\|\mu\| = |\mu|(G)$. The convolution multiplication defined by

$$(\mu * \nu)(A) = \int_G \int_G \chi_A(st) d\mu(s) d\nu(t),$$

where χ_A denotes the characteristic function of the Borel set A , gives $M(G)$ a Banach algebra structure. Again, this space is a Banach lattice algebra. Its identity is the point mass measure on the identity of G .

- (v) Let $\{A_i\}_{i \in I}$ be a family of Banach lattice algebras indexed by a set I , and let $1 \leq p \leq \infty$. Then their ℓ_p -sum $\ell_p(A_i)$, equipped with coordinate-wise order and product, is again a Banach lattice algebra, for if $x = (x_i)$ and $y = (y_i)$ are elements of $\ell_p(A_i)$, then

$$\|xy\|_p \leq \|x\|_p \|y\|_\infty \leq \|x\|_p \|y\|_p.$$

It is clear that $\ell_p(A_i)$ is commutative if and only if A_i is commutative for each $i \in I$. Also, $\ell_\infty(A_i)$ has identity if and only if A_i has identity for each $i \in I$.

We will dive deeper into the properties of Banach lattice algebras in Chapter 3. But before leaving this section, let us quickly mention the relation of Arens products to Banach lattice algebras, for these will play an important role in several of the proofs.

Given Banach spaces A , B and C , and a bounded bilinear map $P: A \times B \rightarrow C$, its *Arens adjoint* is the bounded bilinear map $P^*: C^* \times A \rightarrow B^*$ defined by $P^*(\phi, a)(b) = \phi(P(a, b))$ for every $a \in A$, $b \in B$ and $\phi \in C^*$; when $A = B = C$, its *transpose* is the bounded bilinear map $P^t: A \times A \rightarrow A$ defined

by $P^t(b, a) = P(a, b)$ for all $a, b \in A$. The bilinear map P is called *Arens regular* if $P^{t***t} = P^{***}$. In the particular case that A is a Banach algebra with product $P: A \times A \rightarrow A$, both P^{***} and P^{t***t} define products on A^{**} , called the *first* and *second Arens product*, respectively. When A is a Banach lattice algebra, the first and second Arens product make A^{**} into a Banach lattice algebra with the usual lattice structure (see [HdP84a]). These products extend P , in the sense that the canonical isometry $A \rightarrow A^{**}$ is an algebra homomorphism. When P is Arens regular, the first and second Arens product coincide, and the algebra A is said to be *Arens regular*. If $T: A \rightarrow B$ is a lattice-algebra homomorphism between Banach lattice algebras A and B , its double adjoint $T^{**}: A^{**} \rightarrow B^{**}$ is again a lattice-algebra homomorphism.

1.4 A PRIMER ON f -ALGEBRAS

If we strip a Banach lattice algebra of its norm, we obtain a *lattice-ordered algebra*. Several of the chapters in this thesis are concerned with a particular class of lattice-ordered algebras called f -algebras. The theory of f -algebras has been studied from several points of view. Each of them has its own tools, strengths and weaknesses. In this section we will motivate and introduce f -algebras from three points of view: the algebraic, the operator-theoretic, and the analytic. The contents of this section are based on the monographs [Bir61, Chapter XVII], [Zaa83, Chapter 20], [AB06, Section 2.3], and [DP81].

1.4.1 The algebraic approach

The notions of f -ring and f -algebra were originally introduced by Birkhoff and Pierce [BP56] as part of their seminal work on lattice-ordered rings and algebras. Here we will focus only on lattice-ordered algebras, even though many of the results stated below hold for general lattice-ordered rings.

Definition 1.14. A *partially ordered algebra* is a real algebra A together with a partial order $\leq$ for which, for all $x, y, z \in A$,

- (i) $x \leq y$ implies $x + z \leq y + z$,
- (ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for all $\lambda \in \mathbb{R}_+$,
- (iii) $x, y \geq 0$ implies $xy \geq 0$.

A *lattice-ordered algebra* or ℓ -algebra is a partially ordered algebra which is a lattice under $\leq$ (i.e., given any two elements, their supremum and infimum exist).

Thus a lattice-ordered algebra is in particular a vector lattice.

A subset of a lattice-ordered algebra that is closed under the linear and lattice operations (i.e., a vector sublattice) and the product (i.e., a subalgebra) is called a *sublattice-algebra*. The smallest sublattice-algebra containing a set S is

denoted by $VLA(S)$. A map between lattice-ordered algebras that is both a lattice and an algebra homomorphism is called a *lattice-algebra homomorphism*. As usual, the direct product of lattice-ordered algebras is again a lattice-ordered algebra with the operations and order defined coordinate-wise, and the associated coordinate projections are lattice-algebra homomorphisms. Another important construction, which comes from the theory of universal algebras, is the following.

Definition 1.15. Let $\{A_\gamma\}_\gamma$ be an indexed family of lattice-ordered algebras. Let $\prod_\gamma A_\gamma$ be its direct product, and let $p_\alpha: \prod_\gamma A_\gamma \rightarrow A_\alpha$ be the associated coordinate projections. A *subdirect product* of $\{A_\gamma\}_\gamma$ is a sublattice-algebra A of the direct product $\prod_\gamma A_\gamma$ such that, for every γ and every $x_\gamma \in A_\gamma$, there exists an $x \in A$ with $p_\gamma(x) = x_\gamma$.

The following was originally proved in [BP56] (see also the more easily accessible [Bir61, Section VII.5]).

Theorem 1.16. *A lattice-ordered algebra A is a subdirect product of totally ordered algebras if and only if*

$$a \wedge b = 0 \text{ implies } (ac) \wedge b = 0 = (ca) \wedge b \text{ for all } a, b, c \in A_+. \quad (1.4)$$

Definition 1.17. A lattice-ordered algebra satisfying the conditions of the previous theorem is called an *f-algebra*. We shall also refer to (1.4) as the *f-algebra property*.

The name *f-algebra* stands for “function algebra;” it will become clear in Section 1.4.3 why these structures deserve to be considered as such. Theorem 1.16 is very useful to prove equalities and inequalities in *f-algebras*, for it implies that they only need to be checked in totally ordered algebras.

Proposition 1.18. *Let A be an f-algebra, and let $a, b, c \in A$ with $c \geq 0$. Then the following hold:*

- (i) $c(a \vee b) = (ca) \vee (cb)$ and $(a \vee b)c = (ac) \vee (bc)$;
- (ii) $c(a \wedge b) = (ca) \wedge (cb)$ and $(a \wedge b)c = (ac) \wedge (bc)$;
- (iii) $|ab| = |a||b|$;
- (iv) if $a \wedge b = 0$, then $ab = 0$ (in particular, $a^2 = |a|^2 \geq 0$).

Similarly, one can check that $n|ab - ba| \leq a^2 + b^2$ for every a and b in an *f-algebra* and every $n \in \mathbb{N}$. This readily implies another classical property of *f-algebras*.

Theorem 1.19. *Every Archimedean f-algebra is commutative.*

Another characterization of *f-algebras*, first due to Fuchs [Fuc63, Section IX.3], is the following.

Theorem 1.20. *A lattice-ordered algebra is an f -algebra if and only if all its bands are ring ideals.*

It is straightforward to check that, if an f -algebra A has an algebraic identity e , then e must be positive and a weak order unit. This implies that $a \wedge ne \uparrow a$ for every $a \in A_+$; in fact, the convergence is uniform, as B. de Pagter proved in [DP81, Lemma 10.6].

1.4.2 The operator-theoretic approach

Before presenting further properties of f -algebras, it is necessary to bring operators to the table. This is the approach to f -algebras of [AB06, Section 2.3], the reference we will mainly follow. We start by reconsidering the notions of band preserving operators and orthomorphisms.

Proposition 1.21. *For an operator $T: X \rightarrow X$ on a vector lattice X , the following are equivalent.*

- (i) T is band preserving.
- (ii) $x \perp y$ implies $Tx \perp y$.
- (iii) For each $x \in X$, Tx is contained in the principal band generated by x .

Not every band preserving operator is order bounded. An example, due to M. Meyer, of a non-order bounded band preserving operator can be found in [AB06, Example 2.38].

Definition 1.22. An *orthomorphism* is an order bounded and band preserving operator.

By Proposition 1.21, an orthomorphism $T: X \rightarrow X$ preserves disjointness (i.e., $x \perp y$ implies $Tx \perp Ty$) and therefore every positive orthomorphism is also a lattice homomorphism. Note also that every multiplication operator in an f -algebra is an orthomorphism. Let $\text{Orth}(X)$ denote the set of orthomorphisms on a vector lattice X . The following remarkable result is due independently to Bigard and Keimel [BK69], and Conrad and Diem [CD71].

Theorem 1.23. *Let X be an Archimedean vector lattice. Then $\text{Orth}(X)$, under the pointwise linear operations and order, and the product given by composition, is an Archimedean f -algebra. Moreover, if $T \in \text{Orth}(X)$, then*

$$|T|(|x|) = |T(|x|)| = |Tx| \quad \text{for all } x \in X.$$

In particular, the lattice operations on $\text{Orth}(X)$ can be computed pointwise on positive vectors.

Next we recall some additional properties of orthomorphisms that have immediate implications in the theory of f -algebras.

Theorem 1.24. *Every orthomorphism on an Archimedean vector lattice is order continuous.*

Corollary 1.25. *Multiplication in an Archimedean f -algebra is order continuous.*

Theorem 1.26. *Let X be an Archimedean vector lattice and let $T \in \text{Orth}(X)$. Then*

$$\ker T = [T(X)]^d.$$

In particular, the kernel of an orthomorphism is a band.

Thus, if two orthomorphisms coincide on a set, then they also coincide on the band generated by that set. This was used by Zaanen [Zaa75] to show, without applying Theorem 1.16, that every Archimedean f -algebra is commutative. It can also be used to prove the following.

Corollary 1.27. *Let X be an Archimedean vector lattice and let $e > 0$. Then there exists at most one product on X that makes X an f -algebra having e as its multiplicative unit.*

Let A be an Archimedean f -algebra. For every $a \in A$, the left multiplication operator $L_a: A \rightarrow A$, defined by $L_a(b) = ab$ for all $b \in A$, is an orthomorphism. We therefore have a natural map $L: A \rightarrow \text{Orth}(A)$ given by $L(a) = L_a$. The kernel of this map is the *annihilator* of A :

$$N(A) = \{a \in A : ab = 0 \text{ for all } b \in A\}.$$

The annihilator of an Archimedean f -algebra has a remarkably simple characterization.

Theorem 1.28. *Let A be an Archimedean f -algebra. Its annihilator $N(A)$ is a band, and it satisfies*

$$\begin{aligned} N(A) &= \{a \in A : a^k = 0 \text{ for some } k \in \mathbb{N}\} \\ &= \{a \in A : a^2 = 0\}. \end{aligned}$$

Moreover, if A has the principal projection property, then $N(A)$ is a projection band.

When $N(A) = \{0\}$ (i.e., when $a^2 = 0$ implies $a = 0$), we say that A is *semiprime*. Note that every f -algebra with identity is semiprime. In this case the map $L: A \rightarrow \text{Orth}(A)$ is injective, and moreover $L(A)$ is order dense in $\text{Orth}(A)$ [HdP84a, Proposition 2.1]. Thus we can think of $\text{Orth}(A)$ as a unitarization of A . It is obvious that, if L is surjective, then A has an identity. The converse is also true.

Theorem 1.29. *Let A be an Archimedean f -algebra with identity. Then the map L is bijective, and therefore A is isomorphic to $\text{Orth}(A)$.*

Thus the possible spaces of orthomorphisms of Archimedean vector lattices are precisely the f -algebras with identity. From this theorem it also follows that $\text{Orth}(\text{Orth}(X))$ can be canonically identified with $\text{Orth}(X)$ for every Archimedean vector lattice X .

Recall that every Archimedean vector lattice X has an (essentially unique) order (or Dedekind) completion X^δ . This X^δ is characterized by the fact that it is an order complete vector lattice in which X embeds as an order dense and majorizing vector sublattice. It also has an (essentially unique) universal completion X^u . This X^u is characterized by the fact that it is a universally complete vector lattice (i.e., it is order complete and every set of pairwise disjoint positive vectors has an upper bound) in which X embeds as an order dense sublattice. With the usual identifications, we have $X \subseteq X^\delta \subseteq X^u$.

The vector lattice X^u can be explicitly constructed as a vector lattice of the form $C^\infty(K)$. Here K is an extremally disconnected compact Hausdorff space, and $C^\infty(K)$ is the set of continuous functions $f: K \rightarrow [-\infty, \infty]$ that are finite on a dense set. It can be shown that $C^\infty(K)$ is a universally complete vector lattice under pointwise linear and lattice operations. Moreover, under pointwise multiplication, it is an Archimedean f -algebra with the constant one function as a multiplicative identity. The details of this construction can be found in [LZ71, Section 50].

This illustrates another way in which f -algebras appear naturally when studying vector lattices. And there is more. Since orthomorphisms are order continuous, every positive orthomorphism $T \in \text{Orth}(X)_+$ can be extended uniquely to an orthomorphism, first on X^δ , and then on X^u . Conversely, every orthomorphism on X^u that leaves X invariant restricts to an orthomorphism on X . Thus, with these identifications, we have natural inclusions as sublattice-algebras:

$$\text{Orth}(X) \subseteq \text{Orth}(X^\delta) \subseteq \text{Orth}(X^u).$$

Since X^u is an f -algebra with identity, $\text{Orth}(X^u)$ is precisely the set of multiplication operators by elements of X^u . Hence every $T \in \text{Orth}(X)$ is of the form $T(x) = yx$, for some $y \in X^u$, where the multiplication is taken in X^u . This culminates the intimate relation between orthomorphisms and f -algebras. The omitted details may be found in [AB06, Theorem 2.63].

As an application of these facts, we have the following.

Theorem 1.30 ([AB06, Theorem 2.64]). *Let A be an Archimedean f -algebra with identity. Then A is a sublattice-algebra of A^u .*

In particular, every Archimedean f -algebra with identity can be represented as an f -algebra of extended functions on some extremally disconnected compact Hausdorff space. This remarkable fact leads to our final approach: f -algebras as spaces of functions.

1.4.3 The analytic approach

Consider a family of (extended) real-valued functions on a set. When this family is closed under pointwise addition and scalar multiplication, it is a vector space. When it is also closed under pointwise supremum and infimum, it is an Archimedean vector lattice. Conversely, every abstract Archimedean vector lattice can be realized as a space of extended real-valued functions through its universal completion, as explained above. Suppose that, additionally, this family of functions is closed under pointwise product. Then it is not only a real algebra, but also an f -algebra, as one may readily check. The idea is that the condition defining f -algebras encapsulates the fact that pointwise product preserves supports (this is also closely related to Theorem 1.20).

As it happens for Archimedean vector lattices, abstract Archimedean f -algebras can be realized as concrete spaces of functions. For f -algebras with identity, this was shown in Theorem 1.30 (see also [HJ62] for an alternative construction). For a semiprime Archimedean f -algebra A , we can embed A inside $\text{Orth}(A)$, which is an Archimedean f -algebra with identity, and then apply Theorem 1.30.

In the general case, when A is not necessarily semiprime, we have to consider other products in spaces of functions beyond the pointwise product. The reason is that the pointwise product is always semiprime. For a general A , we have the following.

Theorem 1.31 ([BW17, Corollary 2.5]). *Let A be an Archimedean f -algebra, and let $A^u = C^\infty(K)$ be its universal completion, where K is an extremally disconnected compact Hausdorff space. Identify A as a vector sublattice of $C^\infty(K)$, and let $*$ denote the product in A . Then there exists $w \in C^\infty(K)_+$ such that*

$$(a * b)(t) = w(t)a(t)b(t)$$

for all $t \in K$ for which w , a and b are finite.

1.5 ABOUT f -ALGEBRAS AND NORMS

Once we see f -algebras as spaces of functions, it is natural to add a (complete) norm to this structure, just as it is done for vector lattices (to yield Banach lattices) and for real algebras (to yield Banach algebras). This way we go back to the realm of Banach lattice algebras.

Definition 1.32. A *normed f -algebra* is an f -algebra together with a lattice norm that is submultiplicative. A normed f -algebra is a *Banach f -algebra* if the norm is complete.

Banach f -algebras are first mentioned as such in the work of Martignon [Mar80]. It is surprising that Banach f -algebras have received relatively little attention in the literature, at least compared, on the one hand, to the theory of Banach lattices and algebras and, on the other hand, to the theory of f -algebras

without norms. We hope that by now it is clear to the reader why it is natural to further understand these objects. One of the purposes of this work is to show that not only is this natural, but also interesting, since the interaction between the norm and the f -algebra structure will prove to be rich and non-trivial. Let us first provide some examples of Banach f -algebras.

Example 1.33.

- (i) For every compact Hausdorff space K , the space $C(K)$, with pointwise order and product, is a Banach f -algebra. More generally, if $w \in C(K)_+$ with $\|w\|_\infty \leq 1$, then the product

$$(f * g)(t) = w(t)f(t)g(t), \quad t \in K, \quad f, g \in C(K),$$

turns $C(K)$ into a Banach f -algebra.

- (ii) Any Banach lattice with the zero product is a Banach f -algebra.
- (iii) Let I be any set, and let $w \in \ell_\infty(I)_+$ be such that $\|w\|_\infty \leq 1$. Then for $1 \leq p \leq \infty$, $\ell_p(I)$ with product

$$(x * y)(i) = w(i)x(i)y(i), \quad i \in I, \quad x, y \in \ell_p(I)$$

is a Banach f -algebra.

- (iv) Let H be a Hilbert space, and let D be a commuting set of bounded Hermitian operators on H . Its second commutant D'' is a Banach f -algebra with positive cone the one inherited from $\mathcal{L}(H)$ and composition as product.

The following will be a direct consequence of Theorem 3.1.

Corollary 1.34. *Every Banach f -algebra with identity is lattice-algebra isomorphic to $C(K)$, for some compact Hausdorff space K . If, moreover, the identity has norm 1, then the isomorphism is an isometry.*

In view of this result, one may be led to think that Banach f -algebras are rather trivial. However, in the case they do not have an identity, several elementary questions arise that are surprisingly far from trivial. For instance: given a semiprime normed f -algebra, is its completion (which is a Banach f -algebra) also semiprime? This seemingly innocent question will play a central role in Chapter 8 (see Section 8.4). We will be able to provide in Example 8.79 a semiprime normed f -algebra whose completion is not semiprime.

Biduals are also an important tool to deal with f -algebras. If A is a Banach f -algebra, then we can also endow its bidual A^{**} with either of the two Arens products, as seen in the previous section. Since A is commutative, both Arens products coincide. It was shown by Scheffold [Sch91] that A^{**} , when equipped with the usual lattice structure and the Arens product, becomes again a Banach f -algebra.

1.6 FREE BANACH LATTICES

The study of free objects in Functional Analysis has sparked a great interest in recent years. The free Banach lattice generated by a Banach space [ART18] plays a particularly important role in this thesis. For this reason, we recall its construction and main properties. A detailed exposition of the topic can be found in [OTTT24].

1.6.1 Construction and basic properties

Let E be a Banach space. Let $H[E]$ denote the vector sublattice of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f: E^* \rightarrow \mathbb{R}$ (that is, functions satisfying $f(\lambda x^*) = \lambda f(x^*)$ for all $\lambda \geq 0$ and $x^* \in E^*$). Given $f \in H[E]$, define the quantity

$$\|f\| = \sup \left\{ \sum_{k=1}^n |f(x_k^*)| : n \in \mathbb{N}, x_1^*, \dots, x_n^* \in E^*, \sup_{x \in B_E} \sum_{k=1}^n |x_k^*(x)| \leq 1 \right\}.$$

One can readily check that

$$H_1[E] = \{ f : \|f\| < \infty \}$$

is a sublattice of $H[E]$, and that $\|\cdot\|$ defines a complete lattice norm on $H_1[E]$.

For every $x \in E$, define $\delta_x \in H[E]$ by $\delta_x(x^*) = x^*(x)$ for $x^* \in E^*$. Note that $\|\delta_x\| = \|x\|$ for every $x \in E$. Let $\text{FVL}[E]$ denote the sublattice generated by $\{\delta_x : x \in E\}$ in $H[E]$, and let $\delta_E: E \rightarrow \text{FVL}[E]$ be the linear map defined by $\delta_E(x) = \delta_x$ for all $x \in E$. The space $\text{FVL}[E]$ consists of all expressions that can be built from finitely many elements of the form δ_x using finitely many linear and lattice operations. The pair $(\text{FVL}[E], \delta_E)$ has the universal property of the free vector lattice generated by E : for every linear map $T: E \rightarrow X$ into a vector lattice X , there exists a unique lattice homomorphism $\hat{T}: \text{FVL}[E] \rightarrow X$ such that $T = \hat{T} \circ \delta_E$.

Let $\text{FBL}[E]$ be the norm closure of $\text{FVL}[E]$ in $H_1[E]$. The map δ_E naturally extends to a linear isometry $\phi_E: E \rightarrow \text{FBL}[E]$. The space $\text{FBL}[E]$, together with the map ϕ_E , satisfies the universal property of the free Banach lattice:

Theorem 1.35 ([ART18, Theorem 2.4]). *Let X be a Banach lattice, and let $T: E \rightarrow X$ be an operator. There is a unique lattice homomorphism $\hat{T}: \text{FBL}[E] \rightarrow X$ such that $\hat{T} \circ \phi_E = T$ and $\|\hat{T}\| = \|T\|$.*

The results we will cite from [OTTT24] are always stated in greater generality, because in that paper the object of study is the free p -convex Banach lattice. In this text we will not work with convexity conditions, so our interest will focus on the case $p = 1$.

The elements of $\text{FBL}[E]$ are weak* continuous on bounded sets. Since they are also positively homogeneous, we can isometrically embed $\text{FBL}[E]$ as a sublattice of $C(B_{E^*})$. Most of the time we will regard the elements of $\text{FBL}[E]$ as

functions on B_{E^*} using this identification. Moreover, if E is finite-dimensional, then the restriction of the functions of $\text{FBL}[E]$ to the unit sphere S_{E^*} is onto. Hence, when E is finite-dimensional, the free Banach lattice generated by E can be identified with $C(S_{E^*})$. This is the only case in which the free Banach lattice has a strong unit.

Proposition 1.36 ([OTTT24, Propositions 9.1 and 9.4]). *Let E be a Banach space.*

(i) *E is finite-dimensional if and only if $\text{FBL}[E]$ has a strong unit.*

(ii) *E is separable if and only if $\text{FBL}[E]$ has a quasi-interior point.*

The following are some basic properties of the free Banach lattice.

Proposition 1.37 ([OTTT24, Proposition 2.11]). *Let E be a Banach space.*

(i) *For every $x \in E$, $x \neq 0$, $|\delta_x|$ is a weak order unit in $\text{FBL}[E]$.*

(ii) *If E has dimension strictly greater than one, then $\text{FBL}[E]$ has no non-trivial projection bands.*

(iii) *If E has dimension strictly greater than one, then $\text{FBL}[E]$ is not σ -order complete and has no atoms.*

1.6.2 Operators and their extensions

Every bounded linear operator $T: E \rightarrow F$ between Banach spaces extends uniquely to a lattice homomorphism $\bar{T}: \text{FBL}[E] \rightarrow \text{FBL}[F]$ making the following diagram commute:

$$\begin{array}{ccc} \text{FBL}[E] & \xrightarrow{\bar{T}} & \text{FBL}[F] \\ \phi_E \uparrow & & \uparrow \phi_F \\ E & \xrightarrow{T} & F \end{array}$$

To be more precise, $\bar{T} = \widehat{\phi_F T}$; in particular, $\|\bar{T}\| = \|T\|$. One can even provide an explicit expression of $\bar{T}$.

Lemma 1.38 ([OTTT24, Lemma 3.1]). *Given an operator $T: E \rightarrow F$ between Banach spaces, the extension $\bar{T}: \text{FBL}[E] \rightarrow \text{FBL}[F]$ is given, for $f \in \text{FBL}[E]$, by*

$$\bar{T}(f) = f \circ T^*.$$

There exists a close relation between the properties of T and $\bar{T}$.

Proposition 1.39 ([OTTT24, Proposition 3.2]). *Let $T: E \rightarrow F$ be an operator, and let $\bar{T}: \text{FBL}[E] \rightarrow \text{FBL}[F]$ be its extension. Then*

(i) *T is injective if and only if $\bar{T}$ is injective.*

(ii) *T has dense range if and only if $\bar{T}$ has dense range.*

(iii) T is onto if and only if $\bar{T}$ is onto.

If $T: E \rightarrow F$ is an isomorphism (resp. isometry) between Banach spaces, then $\bar{T}: \text{FBL}[E] \rightarrow \text{FBL}[F]$ is a lattice isomorphism (resp. isometry). Thus isomorphic (resp. isometric) Banach spaces generate isomorphic (resp. isometric) free Banach lattices. Whether the converse holds is an important open question. For the isometric version, the answer is affirmative whenever E and F have smooth duals.

Theorem 1.40 ([OTTT24, Theorem 10.18]). *Let E and F be Banach spaces with smooth dual. A map $T: \text{FBL}[E] \rightarrow \text{FBL}[F]$ is a surjective lattice isometry if and only if $T = \bar{U}$, for some isometry $U: E \rightarrow F$. Consequently, E and F are isometric if and only if $\text{FBL}[E]$ is lattice isometric to $\text{FBL}[F]$.*

This result uses the following description of lattice homomorphisms between free Banach lattices.

Proposition 1.41 ([OTTT24, Proposition 10.1]). *Let E and F be Banach spaces, and let $T: \text{FBL}[E] \rightarrow \text{FBL}[F]$ be a lattice homomorphism. Then there exists a map $\Phi_T: F^* \rightarrow E^*$ such that $Tf = f \circ \Phi_T$ for every $f \in \text{FBL}[E]$. Moreover, Φ_T satisfies the following properties:*

- (i) For every $y^* \in F^*$ and $x \in E$, $\Phi_T(y^*)(x) = (T\delta_x)(y^*)$.
- (ii) Φ_T is positively homogeneous.
- (iii) Φ_T is weak* to weak* continuous on bounded sets.
- (iv) For $y^* \in F^*$, we have $\|\Phi_T y^*\| \leq \|T\| \|y^*\|$.
- (v) For every $y_1^*, \dots, y_m^* \in F^*$ we have

$$\sup_{x \in B_E} \sum_{k=1}^m |\Phi_T(y_k^*)(x)| \leq \|T\| \sup_{y \in B_F} \sum_{k=1}^m |y_k^*(y)|.$$

1.6.3 Projective and injective Banach lattices

Projective Banach lattices were also introduced for the first time in [dPW15].

Definition 1.42. Let $\lambda > 1$ be a real number. A Banach lattice P is λ -projective if whenever X is a Banach lattice, J a closed lattice ideal in X , and $q: X \rightarrow X/J$ the quotient map, then for every lattice homomorphism $T: P \rightarrow X/J$ there is a lattice homomorphism $\hat{T}: P \rightarrow X$ with $T = q\hat{T}$ and $\|\hat{T}\| \leq \lambda\|T\|$. A Banach lattice is *projective* if it is $(1 + \varepsilon)$ -projective for every $\varepsilon > 0$.

Projective Banach lattices can be characterized using free Banach lattices as follows.

Theorem 1.43 ([dPW15, Theorem 10.3]). *A Banach lattice P is projective if and only if for every $\varepsilon > 0$ there exist an index set Γ , a closed sublattice H of $\text{FBL}[\ell_1(\Gamma)]$, a lattice isomorphism $T: P \rightarrow H$ with $\|T\|, \|T^{-1}\| \leq 1 + \varepsilon$, and a lattice homomorphism $R: \text{FBL}[\ell_1(\Gamma)] \rightarrow H$ that is a projection and has norm $\|R\| \leq 1 + \varepsilon$.*

In particular, $\text{FBL}[\ell_1(\Gamma)]$ is projective for every set Γ . Further natural examples of projective Banach lattices are the following.

Theorem 1.44 ([dPW15, Theorem 11.1]). *Every finite-dimensional Banach lattice is projective.*

More generally, we have the following characterization of the $C(K)$ spaces that are projective.

Theorem 1.45 ([AMCRA20, Theorem 1.4]). *Let K be a compact Hausdorff space. Then $C(K)$ is projective if and only if K is an absolute neighbourhood retract.*

The free Banach lattices that are projective must be close to ℓ_1 .

Theorem 1.46 ([AMCRA20, Theorem 1.3]). *Let E be a Banach space. If $\text{FBL}[E]$ is λ -projective, for some $\lambda > 1$, then E has the Schur property.*

The notion dual to projectivity is that of injectivity. However, as already noted in [dPW15, Footnote 1], there are no injective objects in the category of Banach lattices and lattice homomorphisms. For this reason, injective Banach lattices have traditionally been defined with respect to positive operators [Hay77]. More recent work [AT23] has also considered injective objects in the category whose objects are separable Banach lattices and whose arrows are lattice homomorphisms.

2 BAND PROJECTIONS IN SPACES OF REGULAR OPERATORS

David Muñoz Lahoz and Pedro Tradacete, *Band projections in spaces of regular operators*, Trans. Amer. Math. Soc. **377** (2024), no. 7, 5197–5218. MR [4778072](#)

In this chapter we start our study of the algebraic and lattice structures that coexist in $\mathcal{L}_r(X)$, where X is an order complete Banach lattice. We will do so by studying bands and band projections in $\mathcal{L}_r(X)$ exploiting the algebraic structure of the space.

Probably, the most important band in $\mathcal{L}_r(X)$ is the center, which has been extensively studied in the literature (see Section 1.2.3). Other simple band projections on $\mathcal{L}_r(X)$ can be obtained as follows: If P and Q denote band projections on the Banach lattice X , then the operator $T \mapsto PTQ$ also defines a band projection on $\mathcal{L}_r(X)$. These band projections are particularly relevant in the study of components of positive operators: if T is positive, its components of the form PTQ , where P and Q are band projections on X , are called elementary, and are the building blocks of all components (under certain conditions, see [AB06, Section 2.1]).

In the finite dimensional situation, where every linear operator can be identified with multiplication by a matrix, and is automatically bounded and regular, the class of band projections can be easily described. Let $(e_i)_{i=1}^n$ denote a basis of pairwise disjoint positive vectors of a Banach lattice X , so that $\mathcal{L}_r(X)$ is identified with $n \times n$ matrices with respect to this basis. For any set $A \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$ we can define a band projection

$$\begin{aligned} \mathcal{P}_A: \quad \mathcal{L}_r(X) &\longrightarrow \mathcal{L}_r(X) \\ (\alpha_{ij})_{i,j=1}^n &\longmapsto (\alpha_{ij}\chi_A(i,j))_{i,j=1}^n \end{aligned}$$

where $\chi_A(i,j) = 1$ if $(i,j) \in A$ and $\chi_A(i,j) = 0$ elsewhere. Equivalently, if for $1 \leq i \leq n$, P_i denotes the band projection on the band generated by e_i in X , we can write $\mathcal{P}_A(T) = \sum_{(i,j) \in A} P_i T P_j$. It should be clear that all band projections on $\mathcal{L}_r(X)$ when X is finite dimensional are of the above form.

Inner band projections on $\mathcal{L}_r(X)$ will be induced in a similar way as in the finite dimensional case by a band decomposition of the space: Given $\{P_\lambda\}_{\lambda \in \Lambda}$, a family of pairwise disjoint band projections on X , for every subset $\Gamma \subseteq \Lambda \times \Lambda$, the assignment $T \mapsto \bigvee_{(\alpha,\beta) \in \Gamma} P_\alpha T P_\beta$ for $T \geq 0$ can be extended (in a unique way) to a band projection on $\mathcal{L}_r(X)$. Our aim here is to study this family of

band projections and see how they can encode information about the underlying Banach lattice X .

In addition, given that the operator $T \mapsto PTQ$ defines a band projection on $\mathcal{L}_r(X)$ when P and Q are band projections on X , a natural question arises: can band projections on a Banach lattice X be characterized by the multiplication operators they induce in $\mathcal{L}_r(X)$? Recall that given operators A, B on a Banach space E , the multiplication operator $L_AR_B: \mathcal{L}(E) \rightarrow \mathcal{L}(E)$ is defined as $L_AR_B(T) = ATB$. In this respect, a well-known result due to Vala [Val67] asserts that L_AR_B is a compact operator precisely when A and B are compact. Properties of multiplication operators in terms of their symbols have a long history. In particular, one can find results on spectral theory, Fredholm theory, compactness properties, computation of norms, and several others. We refer the interested reader to the survey articles contained in the proceedings volumes [Mat92] and [CM11].

In the setting of regular operators, multiplication operators have been studied by Synnatzschke [Syn80] and Wickstead [Wic15]. With a similar philosophy to Vala's result, we will show that, assuming that X is an order complete Banach lattice, the multiplication operator $L_AR_B: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ is a band projection precisely when A and B are band projections up to a scalar multiple (Theorem 2.35).

In Section 2.1, we start by introducing the notion of inner band projections, which are actually meaningful due to Theorem 2.1. In fact, Section 2.1.1 is devoted to the details for the proof of Theorem 2.1 and some examples. In Section 2.1.2, we show how the inner band projections associated to a pairwise disjoint family of band projections $\{P_\lambda\}_{\lambda \in \Lambda}$ form a Boolean algebra which is isomorphic to $2^{\Lambda \times \Lambda}$ (Proposition 2.10) and study some of their properties. In Section 2.2, we study for which Banach lattices X all band projections on $\mathcal{L}_r(X)$ are inner, and show that this is the case precisely when X is atomic and order continuous (Theorem 2.19). Finally, in Section 2.3, we characterize the multiplication operators $L_AR_B: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ that are band projections.

2.1 INNER BAND PROJECTIONS IN $\mathcal{L}_r(X)$

2.1.1 Construction of inner band projections

The goal of this section is to prove the following result.

Theorem 2.1. *Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . Let $\Gamma \subseteq \Lambda \times \Lambda$ be any subset. Then the map*

$$\begin{aligned} \mathcal{P}_\Gamma: \mathcal{L}_+(X) &\longrightarrow \mathcal{L}_+(X) \\ T &\longmapsto \bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta \end{aligned} \tag{2.1}$$

extends to a unique band projection $\mathcal{P}_\Gamma: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$.

This theorem motivates the following definition.

Definition 2.2. Let X be an order complete Banach lattice. We say that a band projection $\mathcal{P}$ on $\mathcal{L}_r(X)$ is an *inner band projection* if there exists a family of pairwise disjoint band projections $\{P_\lambda\}_{\lambda \in \Lambda}$ on X such that $\mathcal{P} = \mathcal{P}_\Gamma$, as defined in Theorem 2.1, for a certain $\Gamma \subseteq \Lambda \times \Lambda$. We will call *inner projection bands* the bands associated with inner band projections, and we will denote $\mathcal{B}_\Gamma = \text{range}(\mathcal{P}_\Gamma)$.

Theorem 2.1 will be a consequence of the more abstract and general Theorem 3.31. However, we also provide here a more concrete proof using the Riesz–Kantorovich formulae. Even though the present proof is more technical, it also develops some tools that will be used throughout the chapter.

Remark 2.3. Before proceeding to the results, we need to introduce some notation to deal with the indices. Given $\Gamma \subseteq \Lambda \times \Lambda$, where Λ is a set, we will denote the projection onto its first coordinate by

$$\pi_1(\Gamma) = \{ \alpha \in \Lambda : (\alpha, \beta) \in \Gamma \text{ for some } \beta \in \Lambda \},$$

and the projection onto its second coordinate by

$$\pi_2(\Gamma) = \{ \beta \in \Lambda : (\alpha, \beta) \in \Gamma \text{ for some } \alpha \in \Lambda \}.$$

For a fixed $\alpha \in \Lambda$ we will write

$$\Gamma_\alpha = \{ \beta \in \Lambda : (\alpha, \beta) \in \Gamma \},$$

and similarly, for a fixed $\beta \in \Lambda$,

$$\Gamma^\beta = \{ \alpha \in \Lambda : (\alpha, \beta) \in \Gamma \}.$$

Lemma 2.4. Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . Let $\Phi \subseteq \Lambda \times \Lambda$ be a finite subset. Then given $T \in \mathcal{L}_+(X)$:

$$\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta = \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta.$$

Proof. Let $S_1, S_2 \in \mathcal{L}_+(X)$. Given $x \in X_+$ and $\alpha \neq \beta$, the Riesz–Kantorovich formulae give

$$\begin{aligned} 0 &\leq (P_\alpha S_1 \wedge P_\beta S_2)(x) = \inf_{0 \leq u \leq x} P_\alpha(S_1(u)) + P_\beta(S_2(x - u)) \\ &\leq P_\alpha(S_1(x)) \wedge P_\beta(S_2(x)) = 0. \end{aligned}$$

Hence $P_\alpha S_1 \wedge P_\beta S_2 = 0$ whenever $\alpha \neq \beta$, for arbitrary $S_1, S_2 \in \mathcal{L}_+(X)$. Using this, and the fact that the finite sum of pairwise disjoint elements is equal to

their supremum, we have

$$\begin{aligned}
 \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta &= \sum_{\alpha \in \pi_1(\Phi)} P_\alpha T \left(\sum_{\beta \in \Phi_\alpha} P_\beta \right) \\
 &= \bigvee_{\alpha \in \pi_1(\Phi)} P_\alpha T \left(\sum_{\beta \in \Phi_\alpha} P_\beta \right) \\
 &= \bigvee_{\alpha \in \pi_1(\Phi)} P_\alpha T \bigvee_{\beta \in \Phi_\alpha} P_\beta. \tag{2.2}
 \end{aligned}$$

Fixed $\alpha \in \pi_1(\Phi)$, note that for every $x \in X_+$ we can apply again the Riesz–Kantorovich formulae to get

$$\begin{aligned}
 \left(\bigvee_{\beta \in \Phi_\alpha} P_\alpha T P_\beta \right)(x) &= \\
 &= \sup \left\{ \sum_{\beta \in \Phi_\alpha} P_\alpha T P_\beta x_\beta : x_\beta \in X_+ \text{ for } \beta \in \Phi_\alpha \text{ and } \sum_{\beta \in \Phi_\alpha} x_\beta = x \right\} \\
 &= \sup \left\{ P_\alpha T \sum_{\beta \in \Phi_\alpha} P_\beta x_\beta : x_\beta \in X_+ \text{ for } \beta \in \Phi_\alpha \text{ and } \sum_{\beta \in \Phi_\alpha} x_\beta = x \right\} \\
 &\leq P_\alpha T \left(\sup \left\{ \sum_{\beta \in \Phi_\alpha} P_\beta x_\beta : x_\beta \in X_+ \text{ for } \beta \in \Phi_\alpha \text{ and } \sum_{\beta \in \Phi_\alpha} x_\beta = x \right\} \right) \\
 &= P_\alpha T \left(\bigvee_{\beta \in \Phi_\alpha} P_\beta \right)(x),
 \end{aligned}$$

where the inequality is due to the fact that $P_\alpha T \geq 0$. Using this to continue in (2.2) we get

$$\sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta = \bigvee_{\alpha \in \pi_1(\Phi)} P_\alpha T \bigvee_{\beta \in \Phi_\alpha} P_\beta \geq \bigvee_{\alpha \in \pi_1(\Phi)} \bigvee_{\beta \in \Phi_\alpha} P_\alpha T P_\beta = \bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta.$$

To see the reverse inequality, we have to invoke once again the Riesz–Kantorovich formulae for finitely many operators. For $x \in X_+$, we have

$$\begin{aligned}
 \left(\bigvee_{\alpha \in \pi_1(\Phi)} P_\alpha T \bigvee_{\beta \in \Phi_\alpha} P_\beta \right)(x) &= \\
 &= \sup \left\{ \sum_{\alpha \in \pi_1(\Phi)} \left(P_\alpha T \bigvee_{\beta \in \Phi_\alpha} P_\beta \right)(x_\alpha) : x_\alpha \in X_+ \text{ for } \alpha \in \pi_1(\Phi) \text{ and } \sum_{\alpha \in \pi_1(\Phi)} x_\alpha = x \right\}
 \end{aligned}$$

and using that the supremum of finitely many pairwise disjoint elements is equal

to their sum

$$\begin{aligned}
 & \left(\bigvee_{\alpha \in \pi_1(\Phi)} P_\alpha T \bigvee_{\beta \in \Phi_\alpha} P_\beta \right)(x) = \\
 & = \sup \left\{ \sum_{\alpha \in \pi_1(\Phi)} P_\alpha T \sum_{\beta \in \Phi_\alpha} P_\beta x_\alpha : x_\alpha \in X_+ \text{ for } \alpha \in \pi_1(\Phi) \text{ and } \sum_{\alpha \in \pi_1(\Phi)} x_\alpha = x \right\} \\
 & \leq \sup \left\{ \sum_{\alpha \in \pi_1(\Phi)} P_\alpha T \sum_{\beta \in \Phi_\alpha} P_\beta x_{\alpha\beta} : x_{\alpha\beta} \in X_+ \text{ for } (\alpha, \beta) \in \Phi \text{ and } \sum_{(\alpha, \beta) \in \Phi} x_{\alpha\beta} = x \right\} \\
 & = \sup \left\{ \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x_{\alpha\beta} : x_{\alpha\beta} \in X_+ \text{ for } (\alpha, \beta) \in \Phi \text{ and } \sum_{(\alpha, \beta) \in \Phi} x_{\alpha\beta} = x \right\} \\
 & = \left(\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta \right)(x),
 \end{aligned}$$

where the inequality may be seen as follows: let $x_\alpha \in X_+$ for $\alpha \in \pi_1(\Phi)$ be such that $\sum_{\alpha \in \pi_1(\Phi)} x_\alpha = x$, for each $\alpha \in \pi_1(\Phi)$ fix a certain $\beta_0(\alpha) \in \Phi_\alpha$ and define $x_{\alpha\beta} = P_\beta(x_\alpha)$ for $\beta \neq \beta_0(\alpha)$ and $x_{\alpha\beta_0} = x_\alpha - \sum_{\beta \neq \beta_0(\alpha)} P_\beta(x_\alpha)$, then clearly

$$\sum_{(\alpha, \beta) \in \Phi} x_{\alpha\beta} = x \quad \text{and} \quad \sum_{\alpha \in \pi_1(\Phi)} P_\alpha T \sum_{\beta \in \Phi_\alpha} P_\beta(x_{\alpha\beta}) = \sum_{\alpha \in \pi_1(\Phi)} P_\alpha T \sum_{\beta \in \Phi_\alpha} P_\beta(x_\alpha). \quad \square$$

Given a set A , we will denote by $\text{Fin}(A)$ the family of finite subsets of A . Note that $\text{Fin}(A)$ is partially ordered by inclusion, and with this order it is upwards directed.

Lemma 2.5. *Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . Let $\Gamma \subseteq \Lambda \times \Lambda$ be any subset. Then given $T \in \mathcal{L}_+(X)$ and $x \in X_+$:*

$$\left(\bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta \right)(x) = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x.$$

Proof. It is clear that

$$\bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta = \sup_{\Phi \in \text{Fin}(\Gamma)} \bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta.$$

Note that $\{\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta\}_{\Phi \in \text{Fin}(\Gamma)}$ is a net in $\mathcal{L}_r(X)$ bounded by T and increasing, as

$$\bigvee_{(\alpha, \beta) \in \Phi_1} P_\alpha T P_\beta \leq \bigvee_{(\alpha, \beta) \in \Phi_2} P_\alpha T P_\beta \quad \text{whenever } \Phi_1 \subseteq \Phi_2.$$

Since the supremum in $\mathcal{L}_r(X)$ of increasing and bounded nets is taken pointwise, we have that for any given $x \in X_+$,

$$\left(\bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta \right)(x) = \sup_{\Phi \in \text{Fin}(\Gamma)} \left(\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta \right)(x) = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x,$$

where in the last equality we have applied Lemma 2.4. $\square$

Even though the map $\mathcal{P}_\Gamma$ is defined in (2.1) using a supremum, it turns out to be additive:

Lemma 2.6. *Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . Let $\Gamma \subseteq \Lambda \times \Lambda$ be any subset. Then the map $\mathcal{P}_\Gamma: \mathcal{L}_+(X) \rightarrow \mathcal{L}_+(X)$, defined in (2.1), is additive.*

Proof. Given $T, S \in \mathcal{L}_+(X)$, we will check that $\mathcal{P}_\Gamma(T + S) = \mathcal{P}_\Gamma(T) + \mathcal{P}_\Gamma(S)$ by showing the two inequalities. Using that the supremum of the sum is less than or equal to the sum of the suprema, we get the following inequality:

$$\begin{aligned} \mathcal{P}_\Gamma(T + S) &= \bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha(T + S)P_\beta \\ &= \bigvee_{(\alpha, \beta) \in \Gamma} (P_\alpha T P_\beta + P_\alpha S P_\beta) \\ &\leq \bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta + \bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha S P_\beta \\ &= \mathcal{P}_\Gamma(T) + \mathcal{P}_\Gamma(S). \end{aligned}$$

For the reverse inequality, fix $\Phi_1, \Phi_2 \in \text{Fin}(\Gamma)$. Using Lemma 2.4 we have

$$\begin{aligned} \bigvee_{(\alpha, \beta) \in \Phi_1} P_\alpha T P_\beta + \bigvee_{(\alpha, \beta) \in \Phi_2} P_\alpha S P_\beta &\leq \bigvee_{(\alpha, \beta) \in \Phi_1 \cup \Phi_2} P_\alpha T P_\beta + \bigvee_{(\alpha, \beta) \in \Phi_1 \cup \Phi_2} P_\alpha S P_\beta \\ &= \sum_{(\alpha, \beta) \in \Phi_1 \cup \Phi_2} P_\alpha T P_\beta + \sum_{(\alpha, \beta) \in \Phi_1 \cup \Phi_2} P_\alpha S P_\beta \\ &= \sum_{(\alpha, \beta) \in \Phi_1 \cup \Phi_2} P_\alpha(T + S)P_\beta \\ &= \bigvee_{(\alpha, \beta) \in \Phi_1 \cup \Phi_2} P_\alpha(T + S)P_\beta \\ &\leq \mathcal{P}_\Gamma(T + S). \end{aligned}$$

Taking supremum over $\Phi_1 \in \text{Fin}(\Gamma)$ in the previous inequality we get

$$\mathcal{P}_\Gamma(T) + \bigvee_{(\alpha, \beta) \in \Phi_2} P_\alpha S P_\beta \leq \mathcal{P}_\Gamma(T + S).$$

But $\Phi_2 \in \text{Fin}(\Gamma)$ is arbitrary, so taking again supremum over these sets we arrive at $\mathcal{P}_\Gamma(T) + \mathcal{P}_\Gamma(S) \leq \mathcal{P}_\Gamma(T + S)$. $\square$

Now we can present the proof of Theorem 2.1.

Proof of Theorem 2.1. We have shown in Lemma 2.6 that the map $\mathcal{P}_\Gamma: \mathcal{L}_+(X) \rightarrow \mathcal{L}_+(X)$ is additive. Therefore, it extends to a unique positive map, which we shall denote the same way, $\mathcal{P}_\Gamma: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$. From the definition of $\mathcal{P}_\Gamma$ it is clear that $0 \leq \mathcal{P}_\Gamma(T) \leq T$ for any $T \in \mathcal{L}_+(X)$, so

$0 \leq \mathcal{P}_\Gamma \leq \text{id}_{\mathcal{L}_r(X)}$. For $\mathcal{P}_\Gamma$ to be a band projection it is enough to check that $\mathcal{P}_\Gamma \mathcal{P}_\Gamma = \mathcal{P}_\Gamma$.

From $0 \leq \mathcal{P}_\Gamma \leq \text{id}_{\mathcal{L}_r(X)}$ immediately follows $\mathcal{P}_\Gamma \mathcal{P}_\Gamma \leq \mathcal{P}_\Gamma$. It only remains to show the reverse inequality. Fix $\Phi \in \text{Fin}(\Gamma)$, then using Lemmas 2.4 and 2.5 we have that for any $T \in \mathcal{L}_+(X)$ and $x \in X_+$:

$$\begin{aligned}
 \left(\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Gamma(T) P_\beta \right)(x) &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Gamma(T) P_\beta x \\
 &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha \left(\sup_{\Psi \in \text{Fin}(\Gamma)} \sum_{(\gamma, \delta) \in \Psi} P_\gamma T P_\delta P_\beta x \right) \\
 &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha \left(\sup_{\Psi \in \text{Fin}(\Gamma)} \sum_{\gamma \in \Psi^\beta} P_\gamma T P_\beta x \right) \\
 &\geq \sum_{(\alpha, \beta) \in \Phi} \left(\sup_{\Psi \in \text{Fin}(\Gamma)} P_\alpha \sum_{\gamma \in \Psi^\beta} P_\gamma T P_\beta x \right) \\
 &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x = \left(\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta \right)(x).
 \end{aligned}$$

Hence $\bigvee_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Gamma(T) P_\beta \geq \bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta$ for any $\Phi \in \text{Fin}(\Gamma)$. With this we arrive at

$$\begin{aligned}
 \mathcal{P}_\Gamma \mathcal{P}_\Gamma(T) &= \bigvee_{(\alpha, \beta) \in \Gamma} P_\alpha \mathcal{P}_\Gamma(T) P_\beta \\
 &= \sup_{\Phi \in \text{Fin}(\Gamma)} \bigvee_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Gamma(T) P_\beta \\
 &\geq \sup_{\Phi \in \text{Fin}(\Gamma)} \bigvee_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta \\
 &= \mathcal{P}_\Gamma(T),
 \end{aligned}$$

as we wanted to show. $\square$

Remark 2.7 (Order continuous case). Let us consider the case that X has an order continuous norm. Let $\{B_\lambda\}_{\lambda \in \Lambda}$ be a family of bands in X such that $X = \bigoplus_{\lambda \in \Lambda} B_\lambda$; that is, any $x \in X$ can be expressed uniquely as $x = \sum_{\lambda \in \Lambda} P_\lambda x$, where P_λ denotes the band projection onto B_λ , at most countably many of the $P_\lambda x$ are nonzero, and the series converges unconditionally. For example, if A is a maximal family of disjoint positive elements of X , then $X = \bigoplus_{a \in A} B_a$, where B_a is the band generated by $a \in A$.

Fix $\Gamma \subseteq \Lambda \times \Lambda$. For any given $T \in \mathcal{L}_+(X)$ and $x \in X_+$, the set

$$\{ P_\alpha T P_\beta x : (\alpha, \beta) \in \Gamma \}$$

has at most countably many nonzero terms. Let $(a_n)_{n=1}^\infty$ be any enumeration of those terms. Using Lemma 2.5 and the fact that $a_n \geq 0$, we get

$$\begin{aligned} \mathcal{P}_\Gamma(T)(x) &= \left(\bigvee_{(\alpha,\beta) \in \Gamma} P_\alpha T P_\beta \right)(x) \\ &= \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha,\beta) \in \Phi} P_\alpha T P_\beta x \\ &= \sup_{\Phi \in \text{Fin}(\mathbb{N})} \sum_{n \in \Phi} a_n = \sum_{n=1}^\infty a_n = \sum_{(\alpha,\beta) \in \Gamma} P_\alpha T P_\beta x. \end{aligned}$$

We can conclude that, in the order continuous case, inner band projections are computed using infinite sums.

Example 2.8. Let X be a space with a 1-unconditional basis $(e_\lambda)_{\lambda \in \Lambda}$ (with Λ a possibly uncountable set). For every $\lambda \in \Lambda$, consider the band projection

$$\begin{aligned} P_\lambda: X &\longrightarrow X \\ x &\longmapsto x(\lambda)e_\lambda \end{aligned}$$

One can easily check that $\{P_\lambda\}_{\lambda \in \Lambda}$ is a family of pairwise disjoint band projections on X , when X has the Banach lattice structure induced by the basis. So we are under the conditions of Theorem 2.1. Moreover, it is also true that $x = \sum_{\lambda \in \Lambda} P_\lambda x$ for every $x \in X$, in the sense of Remark 2.7. According to this same remark, for each $\Gamma \subseteq \Lambda \times \Lambda$ we have a band projection

$$\begin{aligned} \mathcal{P}_\Gamma: \mathcal{L}_r(X) &\longrightarrow \mathcal{L}_r(X) \\ T &\longmapsto \sum_{(\alpha,\beta) \in \Gamma} P_\alpha T P_\beta \end{aligned}$$

It will follow from Theorem 2.19 that these are all the band projections on $\mathcal{L}_r(X)$.

Example 2.9. Consider the order complete Banach lattice $\ell_\infty(\mathbb{N})$. In this space, the band projections P_i onto the i -th coordinate satisfy the hypotheses of Theorem 2.1. Fix some subset $\Gamma \subseteq \mathbb{N} \times \mathbb{N}$. We want to illustrate with some examples how one can use Lemma 2.5 to compute $\mathcal{P}_\Gamma(T)$ explicitly for certain operators $T \in \mathcal{L}_r(\ell_\infty)$.

Fixed some $a = (a_n)_{n=1}^\infty \in (\ell_\infty)_+$, define the positive operator

$$\begin{aligned} T_a: \ell_\infty &\longrightarrow \ell_\infty \\ x &\longmapsto (a_1 x_1, a_2 x_2, \dots) \end{aligned}$$

Let us denote by $\Delta = \{(n, n) : n \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$ the diagonal of $\mathbb{N} \times \mathbb{N}$. For any $\Phi \in \text{Fin}(\mathbb{N} \times \mathbb{N})$ and $x \in X_+$ we have

$$\sum_{(\alpha,\beta) \in \Phi} P_\alpha T P_\beta x = \sum_{\alpha \in \pi_1(\Phi)} P_\alpha T \sum_{\beta \in \Phi_\alpha} P_\beta x = \begin{cases} a_n x_n & \text{if } n \in \Delta \cap \Phi, \\ 0 & \text{if } n \notin \Delta \cap \Phi, \end{cases}$$

which implies, by Lemma 2.5,

$$\mathcal{P}_\Gamma(T_a)x = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x = \begin{cases} a_n x_n & \text{if } n \in \Delta \cap \Gamma, \\ 0 & \text{if } n \notin \Delta \cap \Gamma. \end{cases}$$

Therefore, $\mathcal{P}_\Gamma(T_a) = T_b$, where $b_n = a_n$ if $n \in \Delta \cap \Gamma$, and zero otherwise.

Now consider the shift operator

$$\begin{aligned} S: \ell_\infty &\longrightarrow \ell_\infty \\ x &\longmapsto (x_2, x_3, \dots) \end{aligned}$$

which is a positive operator. For any $\Phi \in \text{Fin}(\mathbb{N} \times \mathbb{N})$ and $x \in X_+$ we have

$$\sum_{(\alpha, \beta) \in \Phi} P_\alpha S P_\beta x = \begin{cases} x_{n+1} & \text{if } (n, n+1) \in \Phi, \\ 0 & \text{if } (n, n+1) \notin \Phi. \end{cases}$$

and using again Lemma 2.5

$$\mathcal{P}_\Gamma(S)x = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha S P_\beta x = \begin{cases} x_{n+1} & \text{if } (n, n+1) \in \Gamma, \\ 0 & \text{if } (n, n+1) \notin \Gamma. \end{cases}$$

2.1.2 Properties of inner band projections

Now that we have properly defined inner band projections, we can derive some of their properties. The following result shows that inner band projections form a Boolean algebra with the order and lattice operations inherited from the space of regular operators.

Proposition 2.10. *Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . The inner band projections $\{\mathcal{P}_\Gamma : \Gamma \subseteq \Lambda \times \Lambda\}$ (resp. the inner projection bands $\{\mathcal{B}_\Gamma : \Gamma \subseteq \Lambda \times \Lambda\}$) form a Boolean algebra with the same order, suprema and infima as in $\mathfrak{P}(\mathcal{L}_r(X))$ (resp. $\mathfrak{B}(\mathcal{L}_r(X))$). Moreover, this Boolean algebra is isomorphic to $2^{\Lambda \times \Lambda}$ through the map $\Gamma \mapsto \mathcal{P}_\Gamma$ (resp. $\Gamma \mapsto \mathcal{B}_\Gamma$).*

Remark 2.11. Even though inner band projections form a Boolean algebra with the same operations as in $\mathfrak{P}(\mathcal{L}_r(X))$, they do not form, in general, a Boolean subalgebra of $\mathfrak{P}(\mathcal{L}_r(X))$. The reason for this is that the maximal element in the Boolean algebra of inner band projections is $\mathcal{P}_{\Lambda \times \Lambda}$, which need not be the identity, the maximal element in $\mathfrak{P}(\mathcal{L}_r(X))$. This is true even if we assume that $\{P_\lambda\}_{\lambda \in \Lambda}$ form a decomposition of X , so as to have the identity $x = \bigvee_{\lambda \in \Lambda} P_\lambda x$ for every $x \in X_+$.

Let us illustrate this with an example. Consider the Banach lattice ℓ_∞ with the disjoint band projections $\{P_n\}_{n \in \mathbb{N}}$, where P_n is the projection corresponding to the n -th coordinate. We have $x = \bigvee_{n \in \mathbb{N}} P_n x$ for every $x \in X_+$. Let $\phi \in \ell_\infty^*$ be a Banach limit, and consider the positive operator $T = \phi \otimes e_1$, where $e_1 = (1, 0, 0, \dots)$. Then clearly $T P_n = 0$, for every $n \in \mathbb{N}$, so $\mathcal{P}_{\mathbb{N} \times \mathbb{N}}(T) = 0$, while $T \neq 0$. This implies, in particular, that $\mathcal{P}_{\mathbb{N} \times \mathbb{N}}$ is not the identity on $\mathcal{L}_r(X)$.

Proof of Proposition 2.10. It suffices to show the result for the inner band projections. Note first that the minimal and maximal elements in the family of inner band projections are $\mathcal{P}_\emptyset$ and $\mathcal{P}_{\Lambda \times \Lambda}$. As observed before the proposition, $\mathcal{P}_{\Lambda \times \Lambda}$ need not be the identity, which is the maximal element of $\mathfrak{P}(\mathcal{L}_r(X))$.

Fix $\Gamma, \Delta \subseteq \Lambda \times \Lambda$ and let us start by showing that the infimum of two inner band projections $\mathcal{P}_\Gamma \wedge \mathcal{P}_\Delta = \mathcal{P}_\Gamma \mathcal{P}_\Delta$ is given by $\mathcal{P}_{\Gamma \cap \Delta}$. For this, let $\Phi \in \text{Fin}(\Gamma)$, $T \in \mathcal{L}_+(X)$ and $x \in X_+$. Using Lemma 2.5 we have

$$\begin{aligned} \sum_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Delta(T) P_\beta x &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{(\gamma, \delta) \in \Psi} P_\gamma T P_\delta P_\beta x \\ &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{\gamma \in \Psi^\beta} P_\gamma T P_\beta x \\ &= \sum_{(\alpha, \beta) \in \Phi} P_\alpha \left(\sup_{\Psi \in \text{Fin}(\Delta)} \sum_{\gamma \in \Psi^\beta} P_\gamma \right) T P_\beta x, \end{aligned} \quad (2.3)$$

where in the last identity we have used that $\{\sum_{\gamma \in \Psi^\beta} P_\gamma\}_{\Psi \in \text{Fin}(\Delta)}$ is an increasing and bounded net. Note that

$$P_\alpha \left(\sup_{\Psi \in \text{Fin}(\Delta)} \sum_{\gamma \in \Psi^\beta} P_\gamma \right) = P_\alpha \left(\sup_{\Psi \in \text{Fin}(\Delta)} \bigvee_{\gamma \in \Psi^\beta} P_\gamma \right) = P_\alpha \bigvee_{\gamma \in \Delta^\beta} P_\gamma.$$

Using that the projection bands $\{P_\lambda\}_{\lambda \in \Lambda}$ are pairwise disjoint, we can continue in (2.3) to get

$$\sum_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Delta(T) P_\beta x = \sum_{\substack{(\alpha, \beta) \in \Phi \\ \alpha \in \Delta^\beta}} P_\alpha T P_\beta x = \sum_{(\alpha, \beta) \in \Phi \cap \Delta} P_\alpha T P_\beta x.$$

Applying this

$$\begin{aligned} \mathcal{P}_\Gamma(\mathcal{P}_\Delta(T))(x) &= \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha \mathcal{P}_\Delta(T) P_\beta x \\ &= \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi \cap \Delta} P_\alpha T P_\beta x \\ &= \sup_{\Phi \in \text{Fin}(\Gamma \cap \Delta)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x \\ &= \mathcal{P}_{\Gamma \cap \Delta}(T)(x). \end{aligned}$$

For the computations on the complementary and the supremum, we need to show first that $\mathcal{P}_\Gamma + \mathcal{P}_\Delta = \mathcal{P}_{\Gamma \cup \Delta}$ whenever $\Gamma \cap \Delta = \emptyset$. Indeed, take as usual

$T \in \mathcal{L}_+(X)$ and $x \in X_+$, then

$$\begin{aligned} (\mathcal{P}_\Gamma + \mathcal{P}_\Delta)(T)(x) &= \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x + \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{(\gamma, \delta) \in \Psi} P_\gamma T P_\delta x \\ &\geq \sup \left\{ \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x + \sum_{(\gamma, \delta) \in \Psi} P_\gamma T P_\delta x : \Phi \in \text{Fin}(\Gamma), \Psi \in \text{Fin}(\Delta) \right\} \\ &= \sup_{\Phi \in \text{Fin}(\Gamma \cup \Delta)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x = \mathcal{P}_{\Gamma \cup \Delta}(T)x, \end{aligned}$$

where we have first used Lemma 2.5, then that the sum of the suprema is greater than the supremum of the sum, and finally the disjointness of Γ and Δ to be able to apply again Lemma 2.5. To see the reverse inequality, fix first $\Phi \in \text{Fin}(\Gamma)$ and $\Psi \in \text{Fin}(\Delta)$, then we have

$$\sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x + \sum_{(\gamma, \delta) \in \Psi} P_\gamma T P_\delta x \leq \mathcal{P}_{\Gamma \cup \Delta}(T)x,$$

where we are using that $\Phi \cap \Psi = \emptyset$. Since these finite sets were chosen arbitrarily, one can take first supremum on $\Phi \in \text{Fin}(\Gamma)$ and then on $\Psi \in \text{Fin}(\Delta)$ in the previous expression, and thus arrive, according to Lemma 2.5, to

$$\mathcal{P}_\Gamma(T)x + \mathcal{P}_\Delta(T)x \leq \mathcal{P}_{\Gamma \cup \Delta}(T)x.$$

We conclude that, whenever $\Gamma \cap \Delta = \emptyset$, $\mathcal{P}_\Gamma + \mathcal{P}_\Delta = \mathcal{P}_{\Gamma \cup \Delta}$. In particular, taking $\Delta = \bar{\Gamma} = \Lambda \times \Lambda \setminus \Gamma$, we get $\mathcal{P}_{\bar{\Gamma}} = \mathcal{P}_{\Lambda \times \Lambda} - \mathcal{P}_\Gamma$, thus giving the appropriate formula for the complementary.

Next we have to check that the supremum $\mathcal{P}_\Gamma \vee \mathcal{P}_\Delta = \mathcal{P}_\Gamma + \mathcal{P}_\Delta - \mathcal{P}_{\Gamma \cap \Delta}$ is given by $\mathcal{P}_{\Gamma \cup \Delta}$. We have shown this formula in the case that the sets are disjoint. In the general case we can write

$$\mathcal{P}_{\Gamma \cup \Delta} = \mathcal{P}_{\Gamma \cap \bar{\Delta}} + \mathcal{P}_{\Delta \cap \bar{\Gamma}} + \mathcal{P}_{\Gamma \cap \Delta},$$

since $\Gamma \cup \Delta$ is the disjoint union of the sets $\Gamma \cap \bar{\Delta}$, $\Delta \cap \bar{\Gamma}$, and $\Gamma \cap \Delta$. Now using the formulae for the infimum and the complementary

$$\mathcal{P}_{\Gamma \cup \Delta} = \mathcal{P}_\Gamma(\mathcal{P}_{\Lambda \times \Lambda} - \mathcal{P}_\Delta) + \mathcal{P}_\Delta(\mathcal{P}_{\Lambda \times \Lambda} - \mathcal{P}_\Gamma) + \mathcal{P}_\Gamma \mathcal{P}_\Delta = \mathcal{P}_\Gamma + \mathcal{P}_\Delta - \mathcal{P}_{\Gamma \cap \Delta}.$$

From all these computations we can conclude that $\{\mathcal{P}_\Gamma : \Gamma \subseteq \Lambda \times \Lambda\}$ forms a Boolean algebra with the same order, suprema and infima as in $\mathfrak{B}(\mathcal{L}_r(X))$, and also that the map $\Gamma \mapsto \mathcal{P}_\Gamma$ is a Boolean algebra homomorphism. Clearly this map is surjective, so to establish the result it only remains to check that it is injective. Let $\Gamma, \Delta \subseteq \Lambda \times \Lambda$ be different subsets; without loss of generality, we may assume that there exists some $(\lambda, \mu) \in \Gamma \setminus \Delta$. Fix nonzero elements $x_\lambda \in (B_\lambda)_+$ and $x_\mu \in (B_\mu)_+$. Applying Hahn–Banach we can find a positive functional $x^* \in X_+^*$ such that $x^*(x_\mu) = 1$. Then the positive operator $T = x_\lambda \otimes x^*$ satisfies

$$\mathcal{P}_\Gamma(T)(x_\mu) = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x_\mu = x_\lambda \neq 0,$$

whereas

$$\mathcal{P}_\Delta(T)(x_\mu) = \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{(\gamma, \delta) \in \Psi} P_\gamma T P_\delta x_\mu = 0.$$

Hence $\mathcal{P}_\Gamma(T) \neq \mathcal{P}_\Delta(T)$. $\square$

Remark 2.12 (Order continuous case). In the order continuous case, inner band projections do form a Boolean subalgebra of $\mathfrak{P}(\mathcal{L}_r(X))$, as long as the band projections $\{P_\lambda\}_{\lambda \in \Lambda}$ form a decomposition of X . Indeed, according to Remark 2.7, for every $T \in \mathcal{L}_+(X)$ and $x \in X_+$ we have

$$\mathcal{P}_{\Lambda \times \Lambda}(T)x = \sum_{(\alpha, \beta) \in \Lambda \times \Lambda} P_\alpha T P_\beta x = \sum_{\alpha \in \Lambda} P_\alpha T \sum_{\beta \in \Lambda} P_\beta x = \sum_{\alpha \in \Lambda} P_\alpha T x = T x.$$

Hence $\mathcal{P}_{\Lambda \times \Lambda}(T) = T$ for every $T \in \mathcal{L}_+(X)$, so $\mathcal{P}_{\Lambda \times \Lambda} = \text{id}_{\mathcal{L}_r(X)}$. This, together with the previous proposition, implies that inner band projections form a Boolean subalgebra of $\mathfrak{P}(\mathcal{L}_r(X))$.

The following is a partial characterization of inner projection bands. In the order continuous case, this result can be improved to completely characterize these bands.

Proposition 2.13. *Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . Then, for any $\Gamma \subseteq \Lambda \times \Lambda$, the inner projection band $\mathcal{B}_\Gamma$ satisfies*

$$\mathcal{B}_\Gamma \subseteq \{T \in \mathcal{L}_r(X) : T(B_\beta) \subseteq B(\{B_\alpha : \alpha \in \Gamma^\beta\}) \text{ for every } \beta \in \Lambda\}.$$

Proof. Let $T \in \mathcal{B}_\Gamma$ and $x \in (B_\lambda)_+$ for a certain $\lambda \in \Lambda$. Then, using Lemma 2.5,

$$Tx = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta x = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{\alpha \in \Phi^\lambda} P_\alpha T x.$$

Note that for each given $\Phi \in \text{Fin}(\Gamma)$, $\sum_{\alpha \in \Phi^\lambda} P_\alpha T x \in B(\{B_\alpha : \alpha \in \Gamma^\lambda\})$, and since a band is closed under taking arbitrary suprema, the result follows. $\square$

Remark 2.14. The converse inclusion in Proposition 2.13 is not true in general. Consider the operator T presented in Remark 2.11: we have $T(\overline{\text{span}}\{e_n\}) = 0$ for each $n \in \mathbb{N}$. Hence, for any $\Gamma \subseteq \mathbb{N} \times \mathbb{N}$, T satisfies

$$T(\overline{\text{span}}\{e_\beta\}) \subseteq B(\{B_\alpha : \alpha \in \Gamma^\beta\}) \quad \text{for every } \beta \in \mathbb{N}.$$

However, $T \notin \mathcal{B}_\Gamma$ for any $\Gamma \subseteq \mathbb{N} \times \mathbb{N}$, because $\mathcal{P}_\Gamma(T) = 0$.

Remark 2.15 (Order continuous case). In the order continuous case, the other inclusion in Proposition 2.13 does hold, giving the following characterization of inner projection bands:

$$\mathcal{B}_\Gamma = \left\{ T \in \mathcal{L}_r(X) : T(B_\beta) \subseteq \bigoplus_{\alpha \in \Gamma^\beta} B_\alpha \text{ for every } \beta \in \Lambda \right\}.$$

Let us check the converse inclusion to Proposition 2.13. Let $T \in \mathcal{L}_r(X)$ be such that $T(B_\beta) \subseteq \bigoplus_{\alpha \in \Gamma^\beta} B_\alpha$ for every $\beta \in \Lambda$. Then for any given $x \in X$,

$$\mathcal{P}_\Gamma(T)x = \sum_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta x = \sum_{\beta \in \pi_2(\Gamma)} \left(\sum_{\alpha \in \Gamma^\beta} P_\alpha \right) T P_\beta x = \sum_{\beta \in \pi_2(\Gamma)} T P_\beta x,$$

because $P_\beta x \in B_\beta$ and by assumption $T P_\beta x \in \bigoplus_{\alpha \in \Gamma^\beta} B_\alpha$. Now using that $T P_\beta x = 0$ whenever $\beta \notin \pi_2(\Gamma)$, we get

$$\mathcal{P}_\Gamma(T)(x) = \sum_{\beta \in \pi_2(\Gamma)} T P_\beta x = \sum_{\beta \in \Lambda} T P_\beta x = T \left(\sum_{\beta \in \Lambda} P_\beta x \right) = T x.$$

Hence $T = \mathcal{P}_\Gamma(T) \in \mathcal{B}_\Gamma$, as wanted.

Example 2.16. Let us illustrate how one can use previous remark in a particular example. Let $1 \leq p < \infty$, let m denote the Lebesgue measure on $[0, 1]$, and let $X = L_p(m)$. For a fixed $n \in \mathbb{N}$, consider the dyadic partition of the unit interval $[0, 1] = \bigcup_{k=1}^{2^n} I_k^n$, where $I_k^n = [(k-1)/2^n, k/2^n]$ for $k = 1, \dots, 2^n$. Consider the band projections

$$\begin{aligned} P_k: \quad X &\longrightarrow X \\ f &\longmapsto f \chi_{I_k^n}, \end{aligned}$$

which have associated bands

$$B_k = \text{range}(P_k) = \{ f \in X : \text{supp}(f) \subseteq I_k^n \}.$$

Clearly, $X = \bigoplus_{k=1}^{2^n} B_k$. Then, according to Remark 2.7, for any $\Gamma \subseteq \{1, \dots, 2^n\} \times \{1, \dots, 2^n\}$ the operator

$$\begin{aligned} \mathcal{P}_\Gamma: \quad \mathcal{L}_r(X) &\longrightarrow \mathcal{L}_r(X) \\ T &\longmapsto \sum_{(\alpha, \beta) \in \Gamma} P_\alpha T P_\beta \end{aligned}$$

is a band projection.

Set $n = 2$ and $\Gamma = \{(1, 1), (2, 1), (3, 1), (1, 2)\}$. Then, according to Remark 2.15, $T \in \mathcal{B}_\Gamma$ if and only if the following three conditions are satisfied:

- (i) $T(B_1) \subseteq B_1 \oplus B_2 \oplus B_3$, i.e., $\text{supp}(Tf) \subseteq [0, 3/4]$ whenever $\text{supp}(f) \subseteq [0, 1/4]$;
- (ii) $T(B_2) \subseteq B_1$, i.e., $\text{supp}(Tf) \subseteq [0, 1/4]$ whenever $\text{supp}(f) \subseteq [1/4, 1/2]$;
- (iii) $T(B_3) = T(B_4) = 0$, i.e., $Tf = 0$ whenever $\text{supp}(f) \subseteq [1/2, 1]$.

For example, the positive operator $T: X \rightarrow X$ given by

$$(Tf)(x) = f(x/2) \chi_{[0, 1/2]}(x) \quad \text{for every } f \in X \text{ and } x \in [0, 1],$$

satisfies the previous conditions, so $T \in \mathcal{B}_\Gamma$.

For the next property we need first a definition.

Definition 2.17. Let X be a Banach space and let $T, S: X \rightarrow X$ be two operators. We say that T is *left orthogonal* to S or that S is *right orthogonal* to T if $TS = 0$. Given two families of operators $\mathcal{F}$ and $\mathcal{G}$ from X to X , we say that $\mathcal{F}$ is *left orthogonal* to $\mathcal{G}$ and that $\mathcal{G}$ is *right orthogonal* to $\mathcal{F}$ if $TS = 0$ for any $T \in \mathcal{F}$ and $S \in \mathcal{G}$.

Proposition 2.18. Let X be an order complete Banach lattice and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of pairwise disjoint band projections on X . Let $\Gamma, \Delta \subseteq \Lambda \times \Lambda$. Then $\mathcal{B}_\Gamma$ is left orthogonal to $\mathcal{B}_\Delta$ if and only if $\pi_2(\Gamma) \cap \pi_1(\Delta) = \emptyset$.

Proof. Assume $\pi_2(\Gamma) \cap \pi_1(\Delta) = \emptyset$. Let $T \in \mathcal{B}_\Gamma$ and $S \in \mathcal{B}_\Delta$, we first consider the case $T, S \geq 0$. According to Lemma 2.5, for any $x \in X_+$:

$$Sx = \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{(\gamma, \delta) \in \Psi} P_\gamma S P_\delta x \in B(\{B_\gamma : \gamma \in \pi_1(\Delta)\}),$$

and since the projection bands $\{P_\lambda\}_{\lambda \in \Lambda}$ are pairwise disjoint

$$P_\beta Sx = 0 \quad \text{for every } \beta \in \pi_2(\Gamma).$$

But then

$$T(Sx) = \mathcal{P}_\Gamma(T)(Sx) = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} P_\alpha T P_\beta Sx = 0,$$

and since $x \in X_+$ was arbitrary, it follows that $TS = 0$. In the general case, when T and S are not necessarily positive, decomposing into positive and negative parts we get

$$TS = T_+ S_+ - T_- S_+ - T_+ S_- + T_- S_- = 0,$$

because $T_+, T_- \in \mathcal{B}_\Gamma$ and $S_+, S_- \in \mathcal{B}_\Delta$ are all positive.

For the reverse implication, assume $\pi_2(\Gamma) \cap \pi_1(\Delta) \neq \emptyset$, and let us show that $\mathcal{B}_\Gamma$ is not left orthogonal to $\mathcal{B}_\Delta$. Let $\gamma \in \pi_2(\Gamma) \cap \pi_1(\Delta)$, so that there exist $\alpha, \beta \in \Lambda$ with $(\alpha, \gamma) \in \Gamma$ and $(\gamma, \beta) \in \Delta$. Take nonzero elements $x_\alpha \in (B_\alpha)_+$, $x_\gamma \in (B_\gamma)_+$, and $x_\beta \in (B_\beta)_+$. As a consequence of the classical Hahn–Banach theorem, there exist functionals $f, g \in X^*$ such that $f(x_\gamma) = 1$ while $f(y) = 0$ for every $y \notin \overline{\text{span}}\{x_\gamma\}$, and similarly $g(x_\beta) = 1$ while $g(y) = 0$ for every $y \notin \overline{\text{span}}\{x_\beta\}$.

Define the operators $T = f \otimes x_\alpha$ and $S = g \otimes x_\gamma$. Being rank one operators, T and S are regular. Using Lemma 2.5:

$$\begin{aligned} \mathcal{P}_\Delta(S)x_\beta &= \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{(\lambda, \mu) \in \Psi} P_\lambda S P_\mu x_\beta \\ &= \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{(\lambda, \mu) \in \Psi} P_\lambda(x_\gamma) g(P_\mu x_\beta) \\ &= \sup_{\Psi \in \text{Fin}(\Delta)} \sum_{\mu \in \Psi_\gamma} x_\gamma g(P_\mu x_\beta) \\ &= \sup_{\substack{\Psi \in \text{Fin}(\Delta) \\ (\gamma, \beta) \in \Psi}} x_\gamma = x_\gamma \end{aligned}$$

because by assumption $(\gamma, \beta) \in \Delta$. Repeating the same computations

$$\mathcal{P}_\Gamma(T)x_\gamma = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\lambda, \mu) \in \Phi} P_\lambda(x_\alpha) f(P_\mu x_\gamma) = x_\alpha,$$

because $(\alpha, \gamma) \in \Gamma$. In conclusion,

$$\mathcal{P}_\Gamma(T)\mathcal{P}_\Delta(S)x_\beta = x_\alpha \neq 0,$$

so $\mathcal{B}_\Gamma$ is not left orthogonal to $\mathcal{B}_\Delta$. □

2.2 WHEN ARE ALL BAND PROJECTIONS ON $\mathcal{L}_r(X)$ INNER?

In the previous section, we have constructed a particular family of band projections on $\mathcal{L}_r(X)$, called inner band projections. It seems natural to try to characterize the order complete Banach lattices X for which all band projections on $\mathcal{L}_r(X)$ are inner. This is achieved in the following theorem.

Theorem 2.19. *Let X be an order complete Banach lattice. All band projections on $\mathcal{L}_r(X)$ are inner if and only if X is both atomic and order continuous.*

The goal of this section is to prove Theorem 2.19. We do this by splitting the result in several parts: first we show that, if X is not order continuous, there exists a band projection on $\mathcal{L}_r(X)$ that is not inner; then we show that, if X is order continuous but not atomic, there is also a band projection on $\mathcal{L}_r(X)$ that is not inner; finally, we prove that if X is atomic and order continuous, then all band projections on $\mathcal{L}_r(X)$ are inner.

Let us start by recalling a well-known separation result on Banach lattices that will be central in our proofs.

Proposition 2.20. *Let X be a Banach lattice and let I be a closed ideal. If $x_0 \notin I$, $x_0 \geq 0$, there exists a positive functional $x^* \in X_+^*$ such that $x^*|_I = 0$ and $x^*(x_0) = 1$.*

The following procedure to construct band projections on the space of regular operators will also be very useful.

Theorem 2.21. *Let $T: X \rightarrow Y$ be a positive operator between Banach lattices, with Y order complete. For every ideal $I \subseteq X$ the formula*

$$T_I x = \sup\{Ty : 0 \leq y \leq x \text{ and } y \in I\}, \quad \text{for } x \in X_+,$$

defines a positive operator from X to Y . Moreover, the extension of the map

$$\begin{array}{ccc} \mathcal{P}_I: & \mathcal{L}_+(X, Y) & \longrightarrow \mathcal{L}_+(X, Y) \\ & T & \longmapsto T_I \end{array}$$

to the regular operators is a band projection.

Proof. See [AB06, Theorems 1.28 and 2.7]. □

Let us prove first that, when the underlying Banach lattice X is not order continuous, there are band projections on $\mathcal{L}_r(X)$ that are not inner:

Lemma 2.22. *Let X be an order complete Banach lattice. If X is not order continuous, there exists a band projection on $\mathcal{L}_r(X)$ that is not inner.*

Proof. Since X is not order continuous, by [LT79, Proposition 1.a.11] there exists a closed ideal I in X that is not a band. If we denote by $B_I = B(I)$ the band generated by I , then we are saying that $I \neq B_I$. If $\mathcal{P}_I$ is not inner we are done, so we may assume that $\mathcal{P}_I$ is inner. In this case, there exists a family of disjoint bands $\{B_\lambda\}_{\lambda \in \Lambda}$ and some $\Gamma \subseteq \Lambda \times \Lambda$ such that $\mathcal{P}_I = \mathcal{P}_\Gamma$; we may assume, without loss of generality, that the band generated by $\{B_\lambda\}_{\lambda \in \Lambda}$ is X or, in other words, that $x = \bigvee_{\lambda \in \Lambda} P_\lambda x$ for all $x \in X_+$, where P_λ denotes the band projection onto B_λ .

We claim that, for every $\lambda \in \Lambda$, if $\lambda \in \pi_2(\Gamma)$ then $B_\lambda \subseteq I$, while if $\lambda \notin \pi_2(\Gamma)$ then $B_\lambda \cap B_I = \{0\}$. Indeed, suppose that $\lambda_0 \notin \pi_2(\Gamma)$ but $B_{\lambda_0} \cap B_I \neq \{0\}$. Given $x_0 \in B_{\lambda_0} \cap B_I$, $x_0 > 0$, we have

$$\mathcal{P}_I(\text{id}_X)x_0 = P_{B_I}x_0 = x_0 > 0$$

while

$$0 \leq \mathcal{P}_\Gamma(\text{id}_X)x_0 \leq \bigvee_{\lambda \in \Lambda, \lambda \neq \lambda_0} P_\lambda x_0 = 0.$$

This contradicts the fact that $\mathcal{P}_I = \mathcal{P}_\Gamma$. Hence $B_{\lambda_0} \cap B_I = \{0\}$.

If instead $\lambda_0 = \beta_0 \in \pi_2(\Gamma)$, say $(\alpha_0, \beta_0) \in \Gamma$, and we suppose that $B_{\beta_0} \not\subseteq I$, fix $x_0 \in B_{\beta_0} \setminus I$, $x_0 > 0$. By Proposition 2.20 there exists $x^* \in X_+^*$ such that $x^*|_I = 0$ and $x^*(x_0) = 1$. Fix also $y \in B_{\alpha_0}$, $y > 0$, and consider the positive operator $T = x^* \otimes y$. Since $T|_I = 0$, it follows that $\mathcal{P}_I(T) = T_I = 0$, whereas

$$\mathcal{P}_\Gamma(T)x_0 \geq P_{\alpha_0}TP_{\beta_0}x_0 = P_{\alpha_0}(y)x^*(x_0) = y > 0.$$

Hence $\mathcal{P}_\Gamma(T) \neq \mathcal{P}_I(T)$, contradicting the assumption that $\mathcal{P}_\Gamma = \mathcal{P}_I$.

So we must have $B_\lambda \subseteq I$ whenever $B_\lambda \cap B_I \neq \{0\}$. Since I is not equal to B_I , by Proposition 2.20 there exists some $y^* \in X_+^*$ such that $y^*|_I = 0$ while $y^*(x_0) = 1$ for a given $x_0 \in B_I$, $x_0 > 0$. Let $P: X \rightarrow X$ be the band projection onto B_I , and define the positive functional $x^* = P^*y^*$. Then x^* is such that $x^*|_{B_I^d} = 0$ and $x^*|_I = 0$, while $x^*(x_0) = 1$. Pick any $y > 0$ and define the positive operator $T = x^* \otimes y$. Clearly, we have that $TP_\lambda = 0$ for every $\lambda \in \Lambda$ because for every $\lambda \in \Lambda$ either $B_\lambda \subseteq I$ or $B_\lambda \cap B_I = \{0\}$, which implies $B_\lambda \subseteq B_I^d$.

We claim that the band generated by T in $\mathcal{L}_r(X)$, $\mathcal{B}_T$, is not inner. To check this, let $\{C_\xi\}_{\xi \in \Xi}$ be any family of disjoint bands in X , and let $\{Q_\xi\}_{\xi \in \Xi}$ be the corresponding band projections. Suppose that the projection onto $\mathcal{B}_T$ is $\mathcal{P}_\Delta$ for some $\Delta \subseteq \Xi \times \Xi$. Since $\mathcal{B}_T$ is not the zero band, Δ is not empty. Fix some $(\gamma_0, \delta_0) \in \Delta$. Since the family $\{B_\lambda\}_{\lambda \in \Lambda}$ forms a decomposition of X , there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $D = B_{\lambda_1} \cap C_{\delta_0} \neq \{0\}$ and $D' = B_{\lambda_2} \cap C_{\gamma_0} \neq \{0\}$. Fix $y_0 \in D$, $y_0 > 0$, by Proposition 2.20 there exists $f \in X_+^*$ such that $f|_{D^d} = 0$ and $f(y_0) = 1$. Fix $z_0 \in D'$, $z_0 > 0$, and consider the positive operator $S = f \otimes z_0$. Then for every $x \in X_+$:

$$(T \wedge S)(x) = \inf_{0 \leq u \leq x} Tu + S(x - u) = 0, \quad \text{taking } u = P_{\lambda_1} Q_{\delta_0} x,$$

so $S \in \mathcal{B}_T^d$. However

$$\mathcal{P}_\Delta(S)(y_0) \geq Q_{\gamma_0} S Q_{\delta_0} y_0 = Q_{\gamma_0} z_0 = z_0 > 0,$$

which implies $\mathcal{P}_\Delta(S) \neq 0$, a contradiction with the fact that $S \in \mathcal{B}_T^d$. This contradiction arises from the assumption that $\mathcal{P}_\Delta$ is the band projection onto $\mathcal{B}_T$. Therefore, the band projection onto $\mathcal{B}_T$ cannot be inner. $\square$

When the Banach lattice X is order continuous, but not atomic, we can still find band projections on the space of regular operators that are not inner.

Lemma 2.23. *Let X be an order complete Banach lattice. If X is order continuous and not atomic, there exists a band projection on $\mathcal{L}_r(X)$ that is not inner.*

Proof. Since X is order complete, the center of X is the band generated by id_X in $\mathcal{L}_r(X)$ (see [AA02a, Theorem 3.31]). Let $\mathcal{P}_Z$ be the band projection onto the center. We are going to show that $\mathcal{P}_Z$ is not inner.

Let $\{B_\lambda\}_{\lambda \in \Lambda}$ be any decomposition of X into disjoint bands. Since X is not atomic, there exists some $\lambda_0 \in \Lambda$ such that B_{λ_0} is not atomic. We may write

$$B_{\lambda_0} = B_{\lambda_0}^1 \oplus B_{\lambda_0}^2,$$

where $B_{\lambda_0}^2$ is the band generated by the atoms contained in B_{λ_0} and $B_{\lambda_0}^1$ is its complementary; that is, $B_{\lambda_0}^1$ does not contain any atom (of X , since atoms relative to an ideal are also atoms in the whole space). By assumption, $B_{\lambda_0}^1 \neq \{0\}$, so we can pick some $e \in B_{\lambda_0}^1$, $e > 0$. Then B_e is a band in X containing no atoms; let P_e denote the corresponding band projection.

Since B_e is an order continuous Banach lattice with weak unit e , by [LT79, Theorem 1.b.14] there exists a probability space (Ω, Σ, μ) , a (not necessarily closed) ideal $\tilde{X}$ in $L_1(\Omega, \Sigma, \mu)$ and a lattice norm $\|\cdot\|_{\tilde{X}}$ in $\tilde{X}$ such that B_e is order isometric to $(\tilde{X}, \|\cdot\|_{\tilde{X}})$ and we have continuous and dense inclusions

$$L_\infty(\Omega, \Sigma, \mu) \hookrightarrow \tilde{X} \hookrightarrow L_1(\Omega, \Sigma, \mu).$$

Note that, if (Ω, Σ, μ) had an atomic measurable set $A \in \Sigma$, then $\chi_A \in L_\infty(\Omega, \Sigma, \mu) \subseteq \tilde{X}$ would be an atom, contradicting the fact that B_e has no atoms.

Consider the positive operator

$$\begin{aligned} T: L_1(\Omega, \Sigma, \mu) &\longrightarrow L_1(\Omega, \Sigma, \mu) \\ f &\longmapsto \int_\Omega f d\mu \chi_\Omega. \end{aligned}$$

Clearly, we can restrict $T: \tilde{X} \rightarrow \tilde{X}$ and, by means of the above-mentioned order isometry, we can think of this operator as a positive operator $T: B_e \rightarrow B_e$. Define $S = TP_e: X \rightarrow X$. We claim now that $\mathcal{P}_\Gamma \neq \mathcal{P}_Z$ for every $\Gamma \subseteq \Lambda \times \Lambda$. Clearly, if $(\lambda_0, \lambda_0) \notin \Gamma$, $\mathcal{P}_\Gamma(P_e) = 0$ whereas $\mathcal{P}_Z(P_e) = P_e$ (recall that $0 \leq P_e \leq \text{id}_X$). So we must have $(\lambda_0, \lambda_0) \in \Gamma$, in which case $\mathcal{P}_\Gamma(S) = S$; we are going to show that $\mathcal{P}_Z(S) = 0$, thus completing the argument.

For this, we will again think of T as an operator acting on $\tilde{X}$ and in the same way we will identify id_{B_e} with $\text{id}_{\tilde{X}}$. Applying the Riesz–Kantorovich formulae we have that for every $A \in \Sigma$:

$$\begin{aligned} (T \wedge \text{id}_{\tilde{X}})(\chi_A) &= \inf_{0 \leq y \leq \chi_A} Ty + (\chi_A - y) \\ &\leq \inf_{B \subseteq A, B \in \Sigma} T\chi_B + \chi_{A \setminus B} \\ &= \inf_{B \subseteq A, B \in \Sigma} \mu(B)\chi_\Omega + \chi_{A \setminus B}. \end{aligned}$$

We have already observed that the measure space (Ω, Σ, μ) has no atoms, so for every $n \in \mathbb{N}$ there exist pairwise disjoint $B_1, \dots, B_n \in \Sigma$ such that

$$A = \bigcup_{j=1}^n B_j, \quad \text{with } \mu(B_j) = \frac{\mu(A)}{n} \text{ for } j = 1, \dots, n.$$

Applying these sets to the previous inequality gives

$$(T \wedge \text{id}_{\tilde{X}})(\chi_A) \leq \bigwedge_{j=1}^n \left(\frac{\mu(A)}{n} \chi_\Omega + \chi_{A \setminus B_j} \right) = \frac{\mu(A)}{n} \chi_\Omega.$$

Since this inequality holds for every $n \in \mathbb{N}$, it follows that $(T \wedge \text{id}_{\tilde{X}})(\chi_A) = 0$ for every $A \in \Sigma$. But the linear span of the characteristic functions is dense in $L_\infty(\Omega, \Sigma, \mu)$, which is in turn dense in $\tilde{X}$, so $T \wedge \text{id}_{\tilde{X}}$ is zero on $\tilde{X}$. Going back to the space X , this means that $T \wedge \text{id}_{B_e}$ is zero on B_e ; but S is zero outside B_e , and is T when restricted to B_e , so $S \wedge \text{id}_X = 0$. In conclusion, $\mathcal{P}_Z(S) = 0$, and this finishes the proof. $\square$

To complete the proof of Theorem 2.19 it only remains to show that, if the underlying Banach lattice X is atomic and order continuous, then all the band projections on $\mathcal{L}_r(X)$ are inner.

Let X be an order complete Banach lattice and let $a \in X$ be an atom. The band generated by a is $B_a = \overline{\text{span}}\{a\}$. Let P_a be the projection onto B_a . Then for every $x \in X$ we can write $P_a(x) = \lambda_a(x)a$, where $\lambda_a: X \rightarrow \mathbb{R}$ is a positive functional. Banach lattices that are both atomic and order continuous are characterized by the following property.

Theorem 2.24. *Let X be an atomic Banach lattice and let A be a maximal family of pairwise disjoint atoms in X . Then X is order continuous if and only if*

$$x = \sum_{a \in A} \lambda_a(x)a$$

for every $x \in X_+$, where for each x the series has at most countably many nonzero terms and converges unconditionally.

The following is the natural generalization of elementary matrices on $M_n(\mathbb{R})$ (i.e., those with one in a given entry and zeros everywhere else) to general atomic Banach lattices.

Definition 2.25. Let X be an atomic Banach lattice and let A be a maximal family of disjoint atoms. We call *elementary operators* the operators of the form $E_{ab} = \lambda_b \otimes a$, where $a, b \in A$.

Elementary operators have the following properties.

Lemma 2.26. *Let X be an atomic and order complete Banach lattice, and let A be a maximal family of disjoint atoms.*

- (i) *Elementary operators are atoms of $\mathcal{L}_r(X)$.*
- (ii) *If $\mathcal{P}: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ is a band projection, $\mathcal{P}(E_{ab})$ is either E_{ab} or 0 for any $a, b \in A$.*

Proof. (i) Given $a, b \in A$, by definition $a \geq 0$ and also $\lambda_b \geq 0$, so $E_{ab} = \lambda_b \otimes a \geq 0$. Let $0 \leq T \leq E_{ab}$. For every $x \in X_+$ we have

$$\begin{aligned} 0 \leq Tx - TP_b x &= T \left(\bigvee_{\alpha \in A} P_\alpha x - P_b x \right) \\ &= T \left(\bigvee_{\alpha \in A} (P_\alpha x - P_b x) \right) \\ &\leq T \left(\bigvee_{\alpha \in A, \alpha \neq b} P_\alpha x \right) \\ &\leq E_{ab} \left(\bigvee_{\alpha \in A, \alpha \neq b} P_\alpha x \right) = \lambda_b \left(\bigvee_{\alpha \in A, \alpha \neq b} P_\alpha x \right) a = 0. \end{aligned}$$

Therefore $Tx = T(P_b x) = \lambda_b(x)Tb$. But $0 \leq Tb \leq E_{ab}(b) = a$, and since a is an atom, it follows that $Tb = \gamma a$ for a certain $\gamma \in \mathbb{R}_+$. So

$$Tx = \lambda_b(x)\gamma a = \gamma E_{ab}(x) \quad \text{for every } x \in X_+,$$

that is, $T = \gamma E_{ab}$.

- (ii) Since $\mathcal{P}$ is a band projection, $0 \leq \mathcal{P}(E_{ab}) \leq E_{ab}$ for any $a, b \in A$. Since elementary operators are atoms, $\mathcal{P}(E_{ab}) = \gamma E_{ab}$ for a certain $\gamma \in \mathbb{R}_+$. Now using $\mathcal{P}^2 = \mathcal{P}$,

$$\gamma E_{ab} = \mathcal{P}(E_{ab}) = \mathcal{P}(\mathcal{P}(E_{ab})) = \gamma^2 E_{ab},$$

which implies that either $\gamma = 0$ or $\gamma = 1$. □

Remark 2.27. Under the hypotheses of the previous lemma, fix $a, b \in A$. Then for any $T \in \mathcal{L}_+(X)$ and $x \in X_+$ we have:

$$P_a T P_b x = \lambda_b(x) P_a (Tb) = \lambda_b(x) \lambda_a(Tb) a = \lambda_a(Tb) E_{ab} x.$$

Hence $P_a T P_b = \lambda_a(Tb) E_{ab}$. The previous lemma implies that, whenever $\mathcal{P}(E_{ab}) = E_{ab}$, $\mathcal{P}(P_a T P_b) = P_a T P_b$ for every $T \in \mathcal{L}_+(X)$ (and therefore for every $T \in \mathcal{L}_r(X)$), whereas, if $\mathcal{P}(E_{ab}) = 0$, then $\mathcal{P}(P_a T P_b) = 0$ for every $T \in \mathcal{L}_r(X)$.

Lemma 2.28. *Let X be an atomic and order continuous Banach lattice. Then every band projection on $\mathcal{L}_r(X)$ is inner.*

Proof. Let A be a maximal family of pairwise disjoint atoms in X . Let P_a be the band projection onto the band generated by a . We are going to consider inner band projections with respect to the family $\{P_a\}_{a \in A}$. For a given band projection $\mathcal{P}: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$, define

$$\Gamma = \{ (a, b) \in A \times A : \mathcal{P}(E_{ab}) = E_{ab} \}.$$

Note that, using Remark 2.27, we have for any $T \in \mathcal{L}_+(X)$:

$$\mathcal{P}(\mathcal{P}_\Gamma(T)) = \mathcal{P}\left(\bigvee_{(a,b) \in \Gamma} P_a T P_b\right) \geq \bigvee_{(a,b) \in \Gamma} \mathcal{P}(P_a T P_b) = \bigvee_{(a,b) \in \Gamma} P_a T P_b = \mathcal{P}_\Gamma(T).$$

Hence $\mathcal{P} \wedge \mathcal{P}_\Gamma = \mathcal{P} \mathcal{P}_\Gamma \geq \mathcal{P}_\Gamma$, which implies $\mathcal{P}_\Gamma \mathcal{P} = \mathcal{P}_\Gamma$ and also $\mathcal{P} \geq \mathcal{P}_\Gamma$. We are going to show that $\mathcal{P} = \mathcal{P}_\Gamma$. Suppose, on the contrary, that they were different. Then there would exist some $S \in \mathcal{L}_+(X)$ such that $\mathcal{P}(S) > \mathcal{P}_\Gamma(S) = \mathcal{P}_\Gamma(\mathcal{P}(S))$; in other words, $T = \mathcal{P}(S) \in \text{range}(\mathcal{P})$ satisfies $T \geq 0$ and $T - \mathcal{P}_\Gamma(T) > 0$. Let $x \in X_+$ be such that $(T - \mathcal{P}_\Gamma(T))x > 0$, then by continuity and linearity of the operators

$$(T - \mathcal{P}_\Gamma(T))x = \sum_{a \in A} (T - \mathcal{P}_\Gamma(T))P_a x > 0.$$

It follows that there exists some $b \in A$ such that $(T - \mathcal{P}_\Gamma(T))P_b x > 0$. This, in turn, implies that there is some $a \in A$ for which $P_a(T - \mathcal{P}_\Gamma(T))P_b x > 0$. Hence

$$P_a(T - \mathcal{P}_\Gamma(T))P_b > 0. \quad (2.4)$$

If $(a, b) \in \Gamma$, by definition of the inner band projections $P_a \mathcal{P}_\Gamma(T) P_b = P_a T P_b$, contradicting (2.4). Therefore $(a, b) \notin \Gamma$, so $P_a \mathcal{P}_\Gamma(T) P_b = 0$, which by (2.4) gives $P_a T P_b > 0$. But since $0 < P_a T P_b \leq T$, $P_a T P_b$ is also an element of the band range($\mathcal{P}$), which implies $\mathcal{P}(P_a T P_b) = P_a T P_b > 0$. According to the remark after Lemma 2.26, this is equivalent to $\mathcal{P}(E_{ab}) = E_{ab}$, which contradicts the definition of Γ . This contradiction arises from assuming that $\mathcal{P}$ and $\mathcal{P}_\Gamma$ are different, so we must have $\mathcal{P} = \mathcal{P}_\Gamma$. $\square$

From Lemmas 2.22, 2.23 and 2.28 it immediately follows Theorem 2.19. We end this section with a couple of consequences of Theorem 2.19 (although, to be more precise, these are only consequences of Lemma 2.28).

Corollary 2.29. *Let X be an atomic and order continuous Banach lattice. Let A be a maximal family of disjoint atoms in X . Then we have an isomorphism of Boolean algebras*

$$\begin{array}{ccc} 2^{A \times A} & \longrightarrow & \mathfrak{P}(\mathcal{L}_r(X)) \\ \Gamma & \longmapsto & \mathcal{P}_\Gamma \end{array}$$

Proof. By Lemma 2.28, $\mathfrak{P}(\mathcal{L}_r(X))$ contains only the inner band projections that arise from the band projections $\{P_a\}_{a \in A}$ on X . According to Proposition 2.10, the family of inner band projections forms a Boolean algebra that is isomorphic to $2^{A \times A}$ through the isomorphism given. $\square$

Corollary 2.30. *Let X and Y be atomic and order continuous Banach lattices. Let A and B be maximal families of disjoint atoms in X and Y , respectively. If $|A| \neq |B|$, then the Banach lattice algebras $\mathcal{L}_r(X)$ and $\mathcal{L}_r(Y)$ are not isomorphic as Banach lattices.*

Proof. This follows from the fact that the Boolean algebra formed by band projections is invariant under lattice isomorphisms. $\square$

2.3 THE MULTIPLICATION OPERATOR $L_A R_B$

Multiplication operators are defined in the following way.

Definition 2.31. Let X_1, X_2, X_3, X_4 be Banach spaces. Fixed $A \in \mathcal{L}(X_1, X_2)$ and $B \in \mathcal{L}(X_3, X_4)$ we define the *multiplication operator*

$$L_A R_B: \begin{array}{ccc} \mathcal{L}(X_4, X_1) & \longrightarrow & \mathcal{L}(X_3, X_2) \\ T & \longmapsto & ATB \end{array}.$$

These operators have been an extensive object of research in different settings. Here we are going to characterize, assuming that $X = X_1 = X_2 =$

$X_3 = X_4$ is an order complete Banach lattice, when the multiplication operator $L_AR_B: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ is a band projection.

Our results will frequently rely on adaptations of the following observation, which can be found in [MT20].

Remark 2.32. Let X_1, X_2, X_3, X_4 be Banach spaces and let $A \in \mathcal{L}(X_1, X_2)$ and $B \in \mathcal{L}(X_3, X_4)$ be nonzero bounded operators. Consider the multiplication operator $L_AR_B: \mathcal{L}(X_4, X_1) \rightarrow \mathcal{L}(X_3, X_2)$ given by $L_AR_B(T) = ATB$. Choose $x_1 \in X_1$, $x_2^* \in X_2^*$, $x_3 \in X_3$ and $x_4^* \in X_4^*$ such that $x_2^*(Ax_1) = 1 = x_4^*(Bx_3)$. Then the operators

$$\begin{aligned} J_{x_4^*}: X_1 &\longrightarrow \mathcal{L}(X_4, X_1) & J_{x_1}: X_4^* &\longrightarrow \mathcal{L}(X_4, X_1) \\ x &\longmapsto x_4^* \otimes x & x^* &\longmapsto x^* \otimes x_1 \\ \delta_{x_3}: \mathcal{L}(X_3, X_2) &\longrightarrow X_2 & \delta_{x_2^*}: \mathcal{L}(X_3, X_2) &\longrightarrow X_3^* \\ T &\longmapsto Tx_3 & T &\longmapsto T^*x_2^* \end{aligned}$$

make the following diagrams commute

$$\begin{array}{ccc} X_1 & \xrightarrow{A} & X_2 \\ J_{x_4^*} \downarrow & & \uparrow \delta_{x_3} \\ \mathcal{L}(X_4, X_1) & \xrightarrow{L_AR_B} & \mathcal{L}(X_3, X_2) \end{array} \quad \begin{array}{ccc} X_4^* & \xrightarrow{B^*} & X_3^* \\ J_{x_1} \downarrow & & \uparrow \delta_{x_2^*} \\ \mathcal{L}(X_4, X_1) & \xrightarrow{L_AR_B} & \mathcal{L}(X_3, X_2) \end{array} \quad (2.5)$$

Let us start by characterizing when L_AR_B is positive.

Proposition 2.33. *Let X_1, X_2, X_3, X_4 be order complete Banach lattices and let $A \in \mathcal{L}(X_1, X_2)$ and $B \in \mathcal{L}(X_3, X_4)$ be nonzero operators. Then L_AR_B is a positive operator if and only if A and B are both positive or both negative.*

Proof. If A and B are both positive or negative, then clearly L_AR_B is positive.

To see the converse, suppose first that both A and B are not negative. Then there exist $x_3 \in (X_3)_+$ and $x_1 \in (X_1)_+$ such that $(Bx_3)_+ \neq 0$ and $(Ax_1)_+ \neq 0$. By the Hahn–Banach theorem for positive functionals there exists $x_2^* \in (X_2^*)_+$ such that

$$x_2^*((Ax_1)_+) = 1 \text{ and } x_2^*((Ax_1)_-) = 0, \text{ that is, } x_2^*(Ax_1) = 1;$$

and there exists $x_4^* \in (X_4^*)_+$ such that

$$x_4^*((Bx_3)_+) = 1 \text{ and } x_4^*((Bx_3)_-) = 0, \text{ that is, } x_4^*(Bx_3) = 1.$$

It immediately follows that the operators $J_{x_4^*}, J_{x_1}, \delta_{x_3}, \delta_{x_2^*}$ are all positive. Since the composition of positive operators is positive, by (2.5) we have $A \geq 0$ and $B^* \geq 0$, which implies $B \geq 0$.

Now suppose that A is negative. Take $x_1 \in (X_1)_+$ such that $Ax_1 \neq 0$, and let $-x_2^* \in (X_2^*)_+$ be such that $x_2^*(Ax_1) = 1$. Now it is clear that $\delta_{x_2^*} \leq 0$, while $J_{x_1} \geq 0$, so $B^* = \delta_{x_2^*} L_AR_B J_{x_1} \leq 0$, and this again implies $B \leq 0$. Similarly, if B is negative, take $x_3 \in (X_3)_+$ such that $Bx_3 \neq 0$, then there exists $-x_4^* \in (X_4^*)_+$ with $x_4^*(Bx_3) = 1$. It follows that $J_{x_4^*} \leq 0$ while $\delta_{x_3} \geq 0$ so $A = \delta_{x_3} L_AR_B J_{x_4^*} \leq 0$. $\square$

The previous result is stated but not proved in [Syn80, Satz 2.2], although with the further assumption that X_4 is *total*.

If $A, B \in \mathcal{L}_r(X)$ are positive projections, it is not difficult to check that L_AR_B is a positive projection. Naturally, one can ask whether the converse is true. Note that if $\lambda \in \mathbb{R} \setminus \{0\}$, $L_{\lambda A}R_{B/\lambda} = L_AR_B$, but λA and B/λ are no longer positive projections (in general). It turns out, however, that this is the only obstruction.

Proposition 2.34. *Let X be an order complete Banach lattice, and let $A, B \in \mathcal{L}(X)$ be nonzero operators. The multiplication operator $L_AR_B: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ is a positive projection if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that λA and B/λ are positive projections.*

Proof. Suppose first that λA and B/λ are positive projections for a certain $\lambda \in \mathbb{R} \setminus \{0\}$. Then $L_AR_B(T) = (\lambda A)T(B/\lambda) \geq 0$ for any $T \in \mathcal{L}_+(X)$ and

$$\begin{aligned} L_AR_B(L_AR_B(T)) &= AL_AR_B(T)B = A^2TB^2 \\ &= (\lambda A)^2T(B/\lambda)^2 = (\lambda A)T(B/\lambda) = L_AR_B(T). \end{aligned}$$

Hence L_AR_B is a positive projection.

Conversely, if L_AR_B is a positive projection, by the previous proposition A and B are either both positive or negative. Assume first they are both positive. Suppose there exists some $x \in X_+$ such that Bx and B^2x are linearly independent. Then for any $y \in X_+$ there is a $T \in \mathcal{L}_r(X)$ such that $T(Bx) = y$ and $T(B^2x) = y$ (just take $x^* \in X^*$ such that $x^*(Bx) = x^*(B^2x) = 1$, which exists by Hahn–Banach, and set $T = x^* \otimes y$). The fact that $A^2TB^2x = ATBx$ implies $A^2y = Ay$, so $A = A^2$. Since A is not constant (it is a nonzero operator), there are some $y, z \geq 0$ such that $Ay \neq Az$, but then choosing $T \in \mathcal{L}_r(X)$ such that $T(Bx) = y$ and $T(B^2x) = z$ contradicts the fact that $ATBx = ATB^2x$.

So for each $x \in X_+$ with $Bx \neq 0$ there exists some real $\lambda(x) \geq 0$ satisfying $B^2x = \lambda(x)Bx$; in fact, since $Bx = 0$ readily implies $B^2x = 0$, we may assume that this relation holds for any $x \in X$, setting $\lambda(x)$ to be any positive number whenever $Bx = 0$. For any $x, y \in X_+$ we have the following relation:

$$\begin{aligned} \lambda(x+y)(Bx + By) &= \lambda(x+y)B(x+y) = \\ &= B^2(x+y) = B^2x + B^2y = \lambda(x)Bx + \lambda(y)By. \end{aligned}$$

Now we can distinguish two different possibilities:

- (i) If Bx and By are linearly independent, the previous relation readily implies $\lambda(x) = \lambda(x+y) = \lambda(y)$.
- (ii) If Bx and By are linearly dependent, there exist $\mu_1, \mu_2 \in \mathbb{R}_+$, not both zero, such that $\mu_1 Bx + \mu_2 By = 0$. Applying B to this relation we get

$$\mu_1 \lambda(x) Bx + \mu_2 \lambda(y) By = 0.$$

If $\mu_1 \neq 0$, $Bx = -\mu_2/\mu_1 By$, and dividing by μ_1 the previous equation

$$\lambda(x) Bx - \lambda(y) Bx = 0,$$

so $\lambda(x) = \lambda(y)$ whenever $Bx \neq 0$. If $\mu_2 \neq 0$, similarly $\lambda(x) = \lambda(y)$ whenever $By \neq 0$.

So $\lambda(x) = \lambda$ is constant for any $x \in X_+$ with $Bx \neq 0$. We conclude that $B^2 = \lambda B$, i.e., B/λ is a positive projection. Then $\lambda A^2 T B = A T B$, and given $x \in X_+$ with $Bx \neq 0$, for any $y \geq 0$, choosing $T \in \mathcal{L}_r(X)$ such that $T(Bx) = y$ we get from $\lambda A^2 T B x = A T B x$ that $\lambda A^2 y = A y$. Therefore $A^2 = A/\lambda$ and thus λA is a positive projection.

When A and B are both negative, applying the previous argument to $L_{-A} R_{-B} = L_A R_B$ we conclude that there exists some $\mu \geq 0$ such that $(-B)^2 = \mu(-B)$ and $(-A)^2 = 1/\mu(-A)$, so the constant in this case is $\lambda = -\mu$. $\square$

If in the previous proposition we replace positive projections by band projections, everything works.

Theorem 2.35. *Let X be an order complete Banach lattice, and let $A, B \in \mathcal{L}(X)$ be nonzero operators. The multiplication operator $L_A R_B: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ is a band projection if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that λA and B/λ are band projections.*

Proof. If λA and B/λ are band projections, then $L_A R_B = L_{\lambda A} R_{B/\lambda}$ is a positive projection as we have already seen. If $T \in \mathcal{L}_+(X)$:

$$L_A R_B(T) = (\lambda A)T(B/\lambda) \leq \text{id}_X T \text{id}_X = T,$$

which proves $L_A R_B \leq \text{id}_{\mathcal{L}_r(X)}$. It follows that $L_A R_B$ is a band projection.

Conversely, if $L_A R_B$ is a band projection, by the previous proposition there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $P = \lambda A$ and $Q = B/\lambda$ are positive projections. Also, for any $T \in \mathcal{L}_+(X)$,

$$PTQ = L_A R_B(T) \leq \text{id}_{\mathcal{L}_r(X)}(T) = T \quad (2.6)$$

so, multiplying by Q on the right, $PTQ \leq TQ$. This means that the multiplication operator $L_{\text{id}_X - P} R_Q(T) = (\text{id}_X - P)TQ = TQ - PTQ$ is positive, and by Proposition 2.33 this implies that $\text{id}_X - P \geq 0$ (because $Q \geq 0$). So $0 \leq P \leq \text{id}_X$ and $P^2 = P$, that is, P is a band projection. In the same way, multiplying (2.6) on the left by P we get $PTQ \leq PT$, which implies that $L_P R_{\text{id}_X - Q}(T) = PT - PTQ \geq 0$. Applying Proposition 2.33 we conclude that also $Q \leq \text{id}_X$, that is, Q is also a band projection. $\square$

As a consequence, we can characterize band projections in terms of their multiplication operators.

Corollary 2.36. *Let X be an order complete Banach lattice, and let $A \in \mathcal{L}_+(X)$. Then the multiplication operator $L_A R_A$ is a band projection if and only if A is a band projection.*

Proof. Suppose that $L_A R_A$ is a band projection. If $A = 0$ the result follows, so assume $A \neq 0$. By the previous theorem, there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that λA and A/λ are band projections. Since A is positive, λ must also be positive. Using that λA is a projection,

$$\left(\frac{\lambda A}{\lambda^2}\right)^2 = \frac{(\lambda A)^2}{\lambda^4} = \frac{\lambda A}{\lambda^4}.$$

But $\lambda A/\lambda^2 = A/\lambda$ is also a projection, so $(\lambda A/\lambda^2)^2 = A/\lambda$. Join both equations to arrive at $\lambda^2 A = A$. Since $A \neq 0$, it must be the case that $\lambda^2 - 1 = 0$. But $\lambda > 0$ implies $\lambda = 1$, and therefore A is a band projection. $\square$

This result points at an abstract way to study band projections in Banach lattice algebras without identity. This is the idea pursued in the next chapter.

3 BAND PROJECTIONS IN BANACH LATTICE ALGEBRAS

David Muñoz Lahoz, *Band projections and order idempotents in Banach lattice algebras*, J. Math. Anal. Appl. **543** (2025), no. 2, part 3, Paper No. 129045, 27. MR [4824648](#)

The aim of this chapter is to explore results analogous to those of Chapter 2 in the more general context of Banach lattice algebras. Besides contributing to the theory of these structures, this serves the purpose of showing that some properties of spaces of regular operators hold for arbitrary Banach lattice algebras.

Even though the definition of inner band projections uses only the lattice and algebraic operations of $\mathcal{L}_r(X)$, the proof that (2.1) actually defines a band projection relied heavily on the expression of the Riesz–Kantorovich formulae for the lattice operations in $\mathcal{L}_r(X)$. One of the main goals of this chapter is to provide an abstract construction of inner band projections that works on any order complete Banach lattice algebra.

The first step towards an abstract formulation of inner band projections is to determine which elements of a Banach lattice algebra A should play the role of the band projections $\{P_\lambda\}_{\lambda \in \Lambda}$. It is a classical result that an operator $P: X \rightarrow X$ on a Banach lattice X is a band projection if and only if $P^2 = P$ and $0 \leq P \leq \text{id}_X$. When A has an identity e , it seems then natural to regard the elements $p \in A$ satisfying $p^2 = p$ and $0 \leq p \leq e$ as the analogue of band projections. These elements, called order idempotents, were already considered by E. A. Alekhno in the context of ordered Banach algebras (see [Ale12]). For instance, Alekhno showed that the set of order idempotents forms a Boolean algebra, just like the set of band projections on a Banach lattice does. We shall give an alternative proof of this fact (Proposition 3.11), and show that the set of order idempotents is order closed (Proposition 3.12).

But for this we will need to study the ideal generated by the identity element in A . This ideal can be regarded as the abstract analogue of the center of an order complete Banach lattice. We will collect results by L. Martignon [Mar80], C. B. Huijsmans [Hui95] and E. Scheffold [Sch84] to show that this ideal possesses the most fundamental properties of the center: it is lattice and algebra isometric to a space of continuous functions on a compact Hausdorff space, it is a projection band, and it is an inverse closed subalgebra of A .

And what happens if we don't have an identity? Can we still define elements that play the role of band projections? It was proved in Corollary 2.36 that band

projections are characterized by their multiplication operators in the following way: a positive operator $P: X \rightarrow X$ defined on an order complete Banach lattice X is a band projection if and only if the multiplication operator

$$\begin{aligned} L_P R_P: \mathcal{L}_r(X) &\longrightarrow \mathcal{L}_r(X) \\ T &\longmapsto PTP \end{aligned}$$

is a band projection. Band projections could be therefore defined as those elements $p \in A_+$ for which the operator

$$\begin{aligned} L_p R_p: A &\longrightarrow A \\ a &\longmapsto pap \end{aligned}$$

is a band projection. This is analogous to the approach taken by K. Vala in [Val67] to define compact elements in normed algebras.

Several properties of band projections, in this sense, are studied. It is shown that the set of band projections need not be order closed (Example 3.16) or even commute (Remark 3.19); in particular, when we have an identity, band projections and order idempotents may not be the same (Example 3.18). The relation between order idempotents and band projections appears to be subtle, as it is illustrated with several finite-dimensional examples (Examples 3.18 and 3.21). Some conditions guaranteeing that a band projection is order idempotent are also given (Proposition 3.22).

Much better behaved is the set of left and right band projections, defined as those positive elements in A for which both the left and right multiplication operators are band projections. These elements coincide with order idempotents when an identity is present, and they always commute (Remark 3.19). Moreover, they provide the appropriate generalization of band projections that allows for the abstract construction of inner band projections we were looking for. Our main result (Theorem 3.31) is that an analogous formula to (2.1) defines a band projection in any order complete Banach lattice algebra as long as the $\{P_\lambda\}_{\lambda \in \Lambda}$ are taken to be left and right band projections satisfying $P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha$ (where $\delta_{\alpha\beta}$ is the Kronecker delta). To prove it, we cannot resort directly to the Riesz–Kantorovich formulae, as done in the previous chapter; instead, we have to work with the multiplication operators $L_{P_\alpha} R_{P_\beta}$ to translate properties from elements in A to operators acting on A (i.e., to elements in $\mathcal{L}_r(A)$). Even before reaching inner band projections, many of our results will use this technique, that allows for the generalization of properties from spaces of regular operators to Banach lattice algebras. We are not aware of any other systematic use of this idea in the context of Banach lattice algebras, but one can find it already mentioned or implicitly used in some parts of the literature (for instance, in [BH90b, Hui88, Mar80, Wic17a]).

This chapter is organized as follows. In Section 3.1 we study the ideal A_e generated by the identity in a Banach lattice algebra A , showing that it is lattice and algebra isometric to a $C(K)$ space, is a projection band, and is an inverse closed subalgebra of A (Theorem 3.1). In Section 3.2 we introduce order idempotents and band projections in Banach lattice algebras, study

the Boolean algebra structure of $\text{OI}(A)$, and investigate the relation between order idempotents and band projections, presenting several finite-dimensional examples and introducing left and right band projections. Finally, Section 3.3 introduces the complexification of a Banach lattice algebra and uses spectral techniques to further clarify the relation between order idempotents and band projections (Theorem 3.24), culminating in the abstract construction of inner band projections in Section 3.4 (Theorem 3.31).

3.1 THE IDEAL GENERATED BY THE IDENTITY ELEMENT

As explained in Section 1.2.3, the ideal generated by the identity element in $\mathcal{L}_r(X)$, when X is an order complete Banach lattice, is the center of X . It turns out that the notion of the ideal generated by the identity element in Banach lattice algebras has also been studied, and that properties analogous to those in Theorem 1.8 hold in general.

If A is a Banach lattice algebra with identity e , we denote by A_e the (order) ideal generated by e in A . The main properties of this ideal are collected in the following result.

Theorem 3.1. *Let A be a Banach lattice algebra with identity e .*

- (i) *The space $(A_e, \|\cdot\|_e)$, with the order and product of A , is a Banach lattice algebra, lattice and algebra isometric to $C(K)$, for a certain compact Hausdorff K .[†]*
- (ii) *The ideal A_e is a principal projection band in A .*
- (iii) *The space A_e is an inverse closed subalgebra of A .*

The contents of the previous theorem are scattered through the literature: the first assertion follows from the results by L. Martignon in [Mar80], the second assertion is due to E. Scheffold [Sch84, Theorem 3], while the third is mentioned by C. B. Huijsmans in [Hui95] without any further reference. Since this result will be central in our exposition, we devote the rest of this section to present a full proof of it.

To prove the first part of the theorem, we need to talk about Banach lattice algebras that are AM-spaces with unit, in which the order unit coincides with the algebraic identity.

Definition 3.2. Let A be a Banach lattice algebra with identity e . If A is also an AM-space with unit e (i.e., if $A_e = A$ and $\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda e\}$ for every $x \in A$), we say that A is an *AM-algebra with unit e* .

[†]Note that the analogy with Theorem 1.8 is not complete: here we are endowing the ideal generated by the identity with its associated norm, whereas in Theorem 1.8 one works with the operator norm. As it turns out, assumption $\|e\| = 1$ implies that both norms are equal. Yet we need spectra to show this and it will have to wait until Section 3.3. For the time being, it suffices to know that A_e is closed in A . This apparent inconvenience has an advantage: all the contents of this section are valid under the weaker hypothesis $e \geq 0$, which is another common definition of Banach lattice algebras with identity.

The following theorem asserts that AM-algebras with unit are nothing but algebras of continuous functions.

Theorem 3.3. *Every AM-algebra with unit is algebra and lattice isometric to $C(K)$, for a certain compact Hausdorff space K .*

The previous result is a direct consequence of Kakutani's representation theorem for AM-spaces with unit, and the following characterization of the pointwise product in $C(K)$ due to L. Martignon.

Proposition 3.4 ([Mar80, Proposition 1.4]). *Let K be a compact Hausdorff space and let $\star: C(K) \times C(K) \rightarrow C(K)$ be a binary operation such that $(C(K), \star)$ with pointwise order is a Banach lattice algebra. Suppose also that the constant one function is the identity for $\star$. Then $\star$ is the pointwise multiplication.*

The proof of this last result relies heavily on Banach lattice techniques. Here we want to provide an alternative proof of Theorem 3.3 in the style of the Gelfand transform: working with maximal (order and algebraic) ideals and real-valued (lattice and algebra) homomorphisms.

There are two kinds of ideals present in Banach lattice algebras: the ideals as a Banach lattice and the ideals as a Banach algebra. There are also subspaces that are ideals in both senses. There does not seem to be a common terminology for these spaces in the literature. We will refer to ideals in the Banach lattice sense as *order ideals*, and to ideals in the Banach algebra sense as *algebraic ideals* (these, in turn, can be left or right ideals, or both). *Ideals* will be the subspaces that are at the same time an order ideal and a (two-sided) algebraic ideal. The following is a simple characterization of the ideal generated by an element (that is, the intersection of all the ideals containing the element) in a Banach lattice algebra with identity.

Lemma 3.5. *Let A be a Banach lattice algebra with identity and let $a \in A_+$. Then the ideal generated by a is*

$$\{x \in A : |x| \leq rar' \text{ for some } r, r' \in A_+\}.$$

Proof. Call the displayed set I . Let us check that this set is indeed an ideal. If $x, y \in I$, say $|x| \leq rar'$ and $|y| \leq sas'$ for some $r, r', s, s' \in A_+$, then

$$|x + y| \leq |x| + |y| \leq rar' + sas' \leq (r + s)a(r' + s')$$

so $x + y \in I$. Clearly, $-x$ and $|x|$ are elements of I , so I is a sublattice. From the definition it follows that I is an order ideal. To check that it is also an algebraic ideal, let $z \in A$. Then we have

$$|zx| \leq |z||x| \leq (|z|r)ar' \quad \text{and} \quad |xz| \leq |x||z| \leq ra(r'|z|)$$

so both $zx, xz \in I$, and we have thus shown that I is an ideal.

Since A has an identity, $a \in I$. Let J be any ideal of A containing a . Then, since J is a sublattice and an algebraic ideal, $rar' \in J$ for all $r, r' \in A_+$. Since J is an order ideal, $x \in J$ whenever $|x| \leq rar'$. Hence $I \subseteq J$, and so I is the ideal generated by a in A . $\square$

As usual, we say that an ideal in a Banach lattice algebra is *maximal* if it is not contained in any proper ideal. The following is pure routine.

Lemma 3.6. *In a Banach lattice algebra with identity, any proper ideal is contained in a maximal ideal.*

Proof. Let I be a proper ideal in a Banach lattice algebra A with identity e . Let $\mathcal{I}$ be the family of proper ideals in A containing I , ordered by inclusion. Let $\{J_\gamma\}_{\gamma \in \Gamma}$ be a chain in $\mathcal{I}$. Then it is straightforward to check that $J = \bigcup_{\gamma \in \Gamma} J_\gamma$ is again an ideal containing I . Moreover, if $J = A$, then $e \in J$ and so $e \in J_\gamma$ for a certain $\gamma \in \Gamma$. Being J_γ an algebraic ideal, this would imply $J_\gamma = A$, contradicting the definition. So J is proper, and therefore is an upper bound of the chain in $\mathcal{I}$. By Zorn's lemma, there exists a maximal element in $\mathcal{I}$, and this element is a maximal ideal containing I . $\square$

Let A be a Banach lattice algebra and let I be a closed ideal. Perform the usual quotient A/I . Being I a two-sided closed ideal, A/I is again a Banach algebra, and the quotient map $Q: A \rightarrow A/I$ is an algebra homomorphism. Since I is an order ideal, $Q(A_+)$ is a cone in A/I , and the order it induces makes A/I into a Banach lattice, and Q into a lattice homomorphism. It is direct to check that the product of positive elements in A/I is again positive. Hence A/I admits a natural Banach lattice algebra structure, making Q a lattice and algebra homomorphism.

Since the closure of an algebraic ideal is again an algebraic ideal, and the closure of an order ideal is again an order ideal, the closure of an ideal in a Banach lattice algebra is again an ideal. It is then routine to check that maximal ideals in Banach lattice algebras with identity are closed. Moreover, we have the following.

Lemma 3.7. *Let A be an AM-algebra with unit e and let M be a maximal ideal in A . Then A/M is lattice and algebra isomorphic to $\mathbb{R}$.*

Proof. By the discussion preceding the lemma, A/M is a Banach lattice algebra. Suppose that $\dim(A/M) > 1$. Then there would exist two non-zero disjoint elements $Qx, Qy \in A/M$. We are going to show that the ideal I generated by $|Qx|$ in A/M must be proper. Indeed, if $Qy \in I$, by Lemma 3.5 there would exist $r, r' \in A_+$ such that $|Qy| \leq Q(r)|Q(x)|Q(r')$, and then

$$|Qy| = (Q(r)|Q(x)|Q(r')) \wedge |Qy| \leq \|r\| \|r'\| |Q(x)| \wedge |Q(y)| = 0,$$

which is a contradiction. Hence I is proper. It is not difficult to check that $Q^{-1}(I) \subseteq A$ is again an ideal, because Q is both a lattice and algebra homomorphism. Moreover, $M \subseteq Q^{-1}(I) \subsetneq A$, and since $x \in Q^{-1}(I)$, the first

inclusion is proper. This contradicts that M is maximal. So $\dim(A/M)$ must be 1. Then A/M is spanned by Qe and the map

$$\begin{aligned} \mathbb{R} &\longrightarrow A/M \\ \lambda &\longmapsto \lambda Qe \end{aligned}$$

is a lattice and algebra isomorphism. $\square$

Lemma 3.8. *Let A be an AM-algebra with unit e and let $x_0 \in A_+$. If $x_0 \not\leq e$, there exists a lattice and algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ such that $\phi(x_0) > 1$.*

Proof. We have $(x_0 - e)_+ \neq 0$. Let I be the ideal generated by $(x_0 - e)_-$. We claim that $(x_0 - e)_+ \notin I$. Indeed, if it were the case, according to Lemma 3.5 there would exist $r, r' \in A_+$ such that

$$(x_0 - e)_+ \leq r(x_0 - e)_- r' \leq \|r\| \|r'\| (x_0 - e)_-$$

which contradicts the fact that $(x_0 - e)_+$ and $(x_0 - e)_-$ are disjoint. Hence I is proper, and by Lemma 3.6 it is contained in a maximal ideal M . The quotient map associated with M is a lattice and algebra homomorphism $A \rightarrow A/M$. Let $\phi: A \rightarrow \mathbb{R}$ be the composition of this quotient map with the isomorphism between A/M and $\mathbb{R}$ given in Lemma 3.7. This homomorphism satisfies

$$\phi(x_0 - e) = \phi((x_0 - e)_+) > 0,$$

so $\phi(x_0) > \phi(e) = 1$. $\square$

With this set of rather standard tools, we can proceed to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let A be an AM-algebra with unit e . Let K be the set of all lattice and algebra homomorphisms $\phi: A \rightarrow \mathbb{R}$ with $\phi(e) = 1$. We can view K as a subset of $\mathbb{R}^A$. Equip $\mathbb{R}^A$ with the product topology and K with the subspace topology. This topology is Hausdorff, because the product topology is. Note that, for every $\phi \in K$, $|\phi(x)| \leq \phi(\|x\|e) = \|x\|$ holds, which means that $K \subseteq \prod_{x \in A} [-\|x\|, \|x\|]$. By Tychonoff's theorem, the latter product is compact. Moreover, K is closed in this subspace, for if (ϕ_λ) is a net in K pointwise convergent to ϕ , this means that $\phi(x) = \lim \phi_\lambda(x)$ for all $x \in A$, and then by continuity of the algebraic and lattice operations it is easy to check that ϕ is again a lattice and algebra homomorphism with $\phi(e) = 1$. Therefore K is a compact Hausdorff space.

It is useful to note that K has sufficiently many elements. Indeed, for every $x \in A_+$ with $\|x\| > 1$, we have $x \not\leq e$, and by Lemma 3.8 there is some element $\phi \in K$ such that $\phi(x) > 1$.

Now consider the map $T: A \rightarrow C(K)$ where $Tx(\phi) = \phi(x)$ for $x \in A$ and $\phi \in K$. It is easy to check that Tx is indeed continuous. Since operations in $C(K)$ are computed pointwise, and $Te(\phi) = \phi(e) = 1$ for all $\phi \in K$, it follows that T is a lattice and algebra homomorphism that preserves the identity.

We claim $\|Tx\| = \|x\|$ for all $x \in A$. On one hand, $|Tx(\phi)| = |\phi(x)| \leq \|x\|$, so $\|Tx\| \leq \|x\|$. On the other hand, if $\|x\| > 1$, we have seen that there exists some $\phi \in K$ such that $\phi(|x|) > 1$, so that $\|Tx\| \geq |Tx(\phi)| = \phi(|x|) > 1$. If $\|Tx\| < \|x\|$ for some $x \in A$, the element $|x|/\|Tx\|$ would contradict the previous observation. So $\|Tx\| = \|x\|$ for all $x \in A$.

Finally, note that $T(A)$ is a closed subalgebra of $C(K)$ that contains the constants and separates points: if $\phi \neq \psi$ in K , there is some $x \in A$ such that $(Tx)(\phi) = \phi(x) \neq \psi(x) = (Tx)(\psi)$. By the Stone–Weierstrass theorem, $T(A) = C(K)$, and the result is proved. $\square$

Some preliminary remarks before presenting the proof of Theorem 3.1. For the second assertion we will provide an argument similar to [Hui88, Theorems 1 and 2] instead of the original one appearing in [Sch84] because it serves as a first illustration of what will be our main trick in the coming sections, namely, to consider multiplication operators on a Banach lattice algebra, and then exploit the known properties of the space of regular operators. Recall that, on a Banach lattice algebra A , operators of the form

$$\begin{array}{ll} L_a: A \longrightarrow A & R_b: A \longrightarrow A \\ x \longmapsto ax & x \longmapsto xb \end{array},$$

for certain $a, b \in A$, are called *left* and *right multiplication operators*, respectively. Operators of the form $L_a R_b$, for some $a, b \in A$, are called *multiplication operators*.

Proof of Theorem 3.1. (i) We are going to check that $(A_e, \|\cdot\|_e)$ is an AM-algebra with unit e . Then the result will follow from Theorem 3.3. That $(A_e, \|\cdot\|_e)$ is an AM-space with unit e is clear. Moreover, it is a subalgebra, for if $x, y \in A_e$:

$$|xy| \leq |x||y| \leq \|x\|_e \|y\|_e e.$$

From this also follows that $\|xy\|_e \leq \|x\|_e \|y\|_e$, so it is a Banach algebra. Its algebraic identity is obviously e .

(ii) We prove this fact in two steps. First, we show that B_e , the band generated by e , is a projection band. And then we show that $A_e = B_e$.

To prove that B_e is a projection band, it suffices to check that $\sup_n a \wedge ne$ exists for every $a \in A_+$ (in fact, we show a little more, because we will need it later on). Denote $a_n = a \wedge ne$, and observe that for any $m \geq n \geq 1$:

$$(a_m - a_n) \wedge (ne - a_n) = (a_m \wedge ne) - a_n = a_n - a_n = 0.$$

Using the representation of A_e as a $C(K)$ space, and seeing a_m, a_n and ne as continuous functions in K , we are saying that $(a_m - a_n)(t) \wedge (ne - a_n)(t) = 0$ for every $t \in K$. It follows that $a_m - a_n$ and $a_m - n^{-1}a_m a_n = n^{-1}a_m(ne - a_n)$ are also disjoint, so

$$(a_m - a_n) \wedge (a_m - n^{-1}a_m a_n) = 0$$

which implies

$$a_m = a_n \vee (n^{-1}a_m a_n).$$

It follows that

$$0 \leq \|a_m - a_n\| = \|(n^{-1}a_m a_n - a_n)_+\| \leq n^{-1}\|a_m a_n\| \leq n^{-1}\|a\|^2, \quad (3.1)$$

so (a_n) is a Cauchy sequence. It therefore converges to some $b \in A_+$. Since (a_n) is increasing, we must have $b = \sup_n a \wedge ne$, and this shows that B_e is a projection band.

To prove that A_e is a band we need to show that $A_e = B_e$, and for this it suffices to check that $a \in A_e$ whenever $a \in (B_e)_+$. For every $a \in (B_e)_+$, inequality $a \wedge ne \leq ne$ implies that $L_{a \wedge ne}$ is a central operator for every $n \geq 1$. In particular, it is an orthomorphism. If $x, y \in A$ are such that $|x| \wedge |y| = 0$, then $|x| \wedge |(a \wedge ne)y| = 0$ for every $n \geq 1$. But since a is an element of the band generated by e , we have $a = \sup_n a \wedge ne$. Then $a \wedge ne$ converges to a by (3.1). Since both multiplication and lattice operations are continuous for the norm, it follows

$$0 = \lim_{n \rightarrow \infty} |x| \wedge |(a \wedge ne)y| = |x| \wedge |ay|.$$

This shows that L_a is an orthomorphism. By Theorem 1.9, L_a must be a central operator, that is, there exists some $\lambda \in \mathbb{R}_+$ such that $-\lambda \text{id}_A \leq L_a \leq \lambda \text{id}_A$. In particular, $a = L_a(e) \leq \lambda e$, and $a \in A_e$ as wanted.

- (iii) Let $a \in A_e$ and suppose that a^{-1} exists in A . By the previous assertion we can decompose $a^{-1} = a_1 + a_2$ with $a_1 \in A_e$ and $a_2 \in A_e^d$. Then $e = aa^{-1} = aa_1 + aa_2$. Since A_e is a subalgebra, $aa_1 \in A_e$. Also

$$0 \leq |aa_2| \wedge e \leq (|a||a_2|) \wedge e \leq \|a\|_e |a_2| \wedge e = 0,$$

so $aa_2 \in A_e^d$. But $aa_2 = e - aa_1 \in A_e$, so it must be $aa_2 = 0$. Multiplying by a^{-1} we get $a_2 = 0$ and $a^{-1} = a_1 \in A_e$, as wanted. $\square$

Example 3.9. Let A be a Banach lattice algebra with identity e that is an AL-space as a Banach lattice. Then $(A_e, \|\cdot\|_e)$ is an AM-space, but at the same time is closed, so that $(A_e, \|\cdot\|_e)$ is also isomorphic to an AL-space. It follows that A_e must be finite-dimensional. In fact, it will follow from our study of spectra that, under the assumption $\|e\| = 1$, $\|\cdot\| = \|\cdot\|_e$ (see Theorem 3.24). This implies $A_e = \mathbb{R}$. Without the assumption $\|e\| = 1$, however, A_e need not be one-dimensional: just take ℓ_1^n with the coordinate-wise order and product.

3.2 BAND PROJECTIONS AND ORDER IDEMPOTENTS

Band projections play a prominent role in the theory of Banach lattices. These operators are characterized among regular operators by Theorem 1.5. The goal of this section is to define elements in a general Banach lattice algebra that play the same role as band projections do in spaces of regular operators. When an

identity is present, this generalization is straightforward from Theorem 1.5, and has already been considered by E. A. Alekhno in the context of ordered Banach algebras (see [Ale18, Ale12]).

Definition 3.10. Let A be a Banach lattice algebra with identity e . We say that $p \in A$ is *order idempotent* if $p^2 = p$ and $0 \leq p \leq e$. We denote by $OI(A)$ the set of order idempotents in A .

What follows is the generalization of Theorem 1.6. This result appears, with a different proof, in [Ale12, Corollary 2.2].

Proposition 3.11. *Let A be a Banach lattice algebra with identity e . The set of order idempotents $OI(A)$ forms a Boolean algebra with operations*

$$\bar{p} = e - p, \quad p \vee q = p + q - pq, \quad p \wedge q = pq, \quad \text{for } p, q \in OI(A),$$

with zero as the minimum and the identity e as the maximum. Moreover, suprema and infima coincide with those in A .

Proof. By definition, $OI(A)$ is contained in A_e . Represent A_e as a $C(K)$ space using Theorem 3.1. Then $p \in OI(A)$, seen as a continuous function in K , satisfies $p^2 = p$; that is, p is a characteristic function of a clopen set in K . Conversely, any characteristic function of a clopen set is an order idempotent in $C(K)$, and therefore in A . This exhibits an identification of $OI(A)$ with $\{\chi_C : C \subseteq K \text{ clopen}\}$. The result now clearly follows. $\square$

Proposition 3.12. *In a Banach lattice algebra with identity, the set of order idempotents is norm and order closed.*

Proof. Let A be a Banach lattice algebra with identity e . Let (p_n) be a sequence of order idempotents converging to p in norm. Then $p_n^2 = p_n$ and $0 \leq p_n \leq e$ for each $n \in \mathbb{N}$, and since the product is continuous and the positive cone is closed, it follows that p is an order idempotent. This shows that the set of order idempotents is norm closed.

To show that it is also order closed, let (p_α) be a net of order idempotents converging to p in order. The product need not be order continuous, so one has to proceed with more care now. Since the positive cone is always order closed, $0 \leq p \leq e$ follows from $0 \leq p_\alpha \leq e$ for each α . By Theorem 3.1, p and p_α commute, so

$$|p^2 - p_\alpha^2| \leq (p + p_\alpha)|p - p_\alpha| \leq 2|p - p_\alpha|,$$

where the first inequality follows from the submultiplicativity of the modulus. This implies that p_α^2 converges in order to p^2 . Since also $p_\alpha^2 = p_\alpha$ for every α , and the order limit is unique, $p^2 = p$. So p is an order idempotent, and this shows that the set of order idempotents is also order closed. $\square$

The notion of order idempotent seems to abstract some of the most important properties of band projections, but it relies on the existence of an identity element. Of course, this poses no limitation in spaces of regular operators,

where the identity operator is always present. However, this is not the case for general Banach lattice algebras. Can we somehow still abstract the definition and properties of band projections to Banach lattice algebras without identity?

Faced with a similar problem, K. Vala gives in [Val67] a notion of compact and finite rank elements in arbitrary normed algebras as follows: for a normed algebra A , he defines an element a in A to be *compact* (resp. *of finite rank*) if the operator $L_a R_a$ is compact (resp. of finite rank). In view of Corollary 2.36, it makes sense to take Vala's approach and set the following definition.

Definition 3.13. Let A be a Banach lattice algebra. An element $a \in A_+$ is called a *band projection* if $L_a R_a : A \rightarrow A$ is a band projection. The set of band projections in A is denoted

$$BP(A) = \{a \in A_+ : L_a R_a \text{ is a band projection}\}.$$

It is also convenient to have the following notions.

Definition 3.14. Let A be a Banach lattice algebra. An element $a \in A_+$ is called a *left band projection* (resp. *right band projection*) if L_a (resp. R_a) is a band projection. The sets of left and right band projections in A are denoted

$$BP_l(A) = \{a \in A_+ : L_a \text{ is a band projection}\},$$

$$BP_r(A) = \{a \in A_+ : R_a \text{ is a band projection}\}.$$

Remark 3.15. Let A be a Banach lattice algebra. According to Theorem 1.5, $a \in A_+$ is a band projection if and only if

$$0 \leq axa \leq x \text{ and } a^2 x a^2 = axa \text{ for all } x \in A_+.$$

Similarly, $a \in A_+$ is a left band projection if and only if

$$0 \leq ax \leq x \text{ and } a^2 x = ax \text{ for all } x \in A_+,$$

and a right band projection if and only if

$$0 \leq xa \leq x \text{ and } xa^2 = xa \text{ for all } x \in A_+.$$

In particular, the sets $BP(A)$, $BP_l(A)$ and $BP_r(A)$ are all norm closed.

If the multiplication in A is jointly order continuous, it follows from the previous remark that $BP(A)$, $BP_l(A)$ and $BP_r(A)$ are order closed. In general, however, this will not be true, as the following example shows.

Example 3.16. Consider the Banach lattice c of convergent real sequences, with the product defined by

$$x \star y = \left(\lim_{n \rightarrow \infty} x_n \right) \left(\lim_{n \rightarrow \infty} y_n \right) e \text{ where } e = (1, 1, 1, \dots).$$

Associativity of this product follows from

$$\lim_{n \rightarrow \infty} (x \star y)_n = \left(\lim_{n \rightarrow \infty} x_n \right) \left(\lim_{n \rightarrow \infty} y_n \right),$$

bilinearity is clear, and it is not difficult to check that, under the usual supremum norm, c with $\star$ is a Banach algebra. Moreover, the product of positive elements is positive, so c is actually a Banach lattice algebra (without identity).

It is clear that $c_0 = BP(c) = BP_l(c) = BP_r(c)$. In particular, $x_n = \sum_{i=1}^n e_i$, where e_i denotes the i th coordinate vector, is a (left and right) band projection. Also, $x_n \uparrow e$. However, e is not a (left or right) band projection, because for $x = e - e_1$ we have

$$e \star x \star e = e \not\leq x.$$

It is natural to ask what is the relation between the sets $BP(A)$, $BP_l(A)$, $BP_r(A)$ and, in the case with identity, $OI(A)$. From the definition, it should be clear that $BP_l(A) \cap BP_r(A) \subseteq BP(A)$, and that, in the case with identity, $BP_l(A) = OI(A) = BP_r(A)$ (for $p \in BP_l(A)$, just evaluate L_p at the identity element, and similarly for $p \in BP_r(A)$). In particular, $OI(A) = BP_l(A) \cap BP_r(A) \subseteq BP(A)$. Is it true that $OI(A) = BP(A)$? This seems to be the case at least for the classical examples of Banach lattice algebras.

Example 3.17. (i) Let X be an order complete Banach lattice. If $P \in BP(\mathcal{L}_r(X))$, by Corollary 2.36, P is a band projection, that is, $P \in OI(\mathcal{L}_r(X))$.

(ii) Let $f \in BP(C(K))$, for a certain compact Hausdorff space K , and let $\mathbb{1}_K$ be the constant one function in $C(K)$. Then $L_f R_f(\mathbb{1}_K) = (L_f R_f)^2(\mathbb{1}_K)$ implies $f^2 = f^4$, so f must be the characteristic function of a clopen set. It follows that f is an order idempotent.

(iii) Let G be a locally compact group and consider the Banach lattice algebra $M(G)$. Since the identity δ_e is an atom, if $\mu \in OI(M(G))$, inequality $0 \leq \mu \leq \delta_e$ implies $\mu = \lambda \delta_e$ for some $\lambda \geq 0$. Then $\mu^2 = \mu$ (we denote $\mu^2 = \mu * \mu$) forces either $\lambda = 1$ or $\lambda = 0$. Hence $OI(M(G)) = \{0, \delta_e\}$ is trivial.

We are going to show that $BP(M(G))$ is also trivial. If $\mu \in BP(M(G))$, evaluating $L_\mu R_\mu$ at δ_e yields that μ^2 must be an order idempotent. That is, either $\mu^2 = 0$ or $\mu^2 = \delta_e$. By direct computation

$$\mu^2(G) = \iint_{G \times G} \chi_G(st) d\mu(s) d\mu(t) = \int_G \mu(s^{-1}G) d\mu(s) = \mu(G)^2,$$

so if $\mu^2 = 0$, $\mu(G)^2 = 0$, and since $\mu \geq 0$, this implies $\mu = 0$. Now suppose $\mu \neq 0$, that is, $\mu^2 = \delta_e$. We can decompose $\mu = \mu_d + \mu_c$, where $\mu_d \geq 0$ is discrete (i.e., is contained in the band generated by the atomic measures, which is precisely the closed linear span of said measures, because $M(G)$ is order continuous) and $\mu_c \geq 0$ is continuous (i.e., is contained in the disjoint complement). Since

$$\mu_c^2(\{g\}) = \int_G \mu_c(\{s^{-1}g\}) d\mu_c(s) = 0 \quad \text{for every } g \in G,$$

the measure μ_c^2 cannot have any discrete component. However,

$$\delta_e = \mu^2 = \mu_d^2 + \mu_c^2 + \mu_d * \mu_c + \mu_c * \mu_d$$

and since every measure involved is positive, we must have $\mu_c^2 = 0$. Equality $\mu_c(G)^2 = \mu_c^2(G) = 0$ implies $\mu_c = 0$ and so μ is discrete. Write

$$\mu = \sum_{g \in G} \lambda_g \delta_g$$

for some $\lambda_g \in \mathbb{R}_+$. By definition,

$$\delta_e = \mu^2 = \sum_{g \in G} \left(\sum_{h \in G} \lambda_h \lambda_{h^{-1}g} \right) \delta_g,$$

and since the λ_h are positive, this can only happen if $\lambda_h = 0$ for all $h \neq e$. Therefore $\mu = \lambda_e \delta_e$, and from $\mu^2 = \mu^4$ it follows that $\lambda_e = 1$. We have thus shown that if $\mu \in BP(M(G))$ is different from 0, it must be δ_e , as wanted.

- (iv) A. W. Wickstead provided in [Wic17c] an explicit list of all unital two-dimensional Banach lattice algebras. One can check, by means of direct computation, that the order idempotents coincide with the band projections in all two-dimensional Banach lattice algebras.

Contrary to this evidence, it is not true that in Banach lattice algebras with identity all band projections are order idempotents, and an example already exists in dimension 3.

Example 3.18. Consider $\mathbb{R}^2$ with the usual lattice structure and any lattice norm, and let $\mathcal{L}_r(\mathbb{R}^2)$ be its space of regular operators. The space $\mathcal{L}_r(\mathbb{R}^2)$ is a Banach lattice algebra, which we may identify with $M_2(\mathbb{R})$, the space of 2×2 real matrices, where the order is defined entry-wise and the product is the composition (that is, the usual matrix product). Consider the set of upper triangular matrices

$$\begin{aligned} A &= \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}. \end{aligned}$$

Being spanned by disjoint elements, A is a 3-dimensional sublattice of $M_2(\mathbb{R})$. Moreover, one can easily check that the product of upper-triangular matrices is again upper-triangular. Hence, A is a Banach lattice algebra with identity when equipped with the order, product, and norm inherited from $M_2(\mathbb{R})$.

Now consider $p = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly $p \geq 0$ and, for all $a, b, c \in \mathbb{R}$,

$$L_p R_p \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $L_p R_p = 0$; in particular, $p \in BP(A)$. However, $p \not\leq \text{id}_{\mathbb{R}^2}$, so $p \notin OI(A)$.

Let us compute exactly $BP(A)$ and $OI(A)$. It is direct to check that

$$OI(A) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Now let $\bar{p} = \begin{pmatrix} p_1 & p_2 \\ 0 & p_3 \end{pmatrix} \in BP(A)$. For any $a, b, c \in \mathbb{R}_+$:

$$\begin{pmatrix} p_1 & p_2 \\ 0 & p_3 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ 0 & p_3 \end{pmatrix} = \begin{pmatrix} p_1^2 a & p_1 p_2 a + p_1 p_3 b + p_2 p_3 c \\ 0 & c p_3^2 \end{pmatrix} \leq \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Setting $b = 0$ we get the inequality in the upper right entry

$$p_2(ap_1 + cp_3) \leq 0.$$

Since a and c can take any positive value, and p_1, p_2, p_3 are positive, either $p_2 = 0$ or $p_1 = p_3 = 0$.

- (i) If $p_2 = 0$, equality $\bar{p}^2 = \bar{p}^4$ implies $p_1^2 = p_1^4$, so $p_1 = 0$ or $p_1 = 1$, and similarly $p_3 = 0$ or $p_3 = 1$. The only band projections in this case are the order idempotents.
- (ii) If $p_2 \neq 0$, then $p_1 = p_3 = 0$, and $\bar{p}$ is a band projection for any $p_2 \geq 0$.

Putting together these computations:

$$BP(A) = OI(A) \cup \left\{ \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} : p \in \mathbb{R}_+ \right\}.$$

Remark 3.19. This example already illustrates a difference between $BP(A)$ and $OI(A)$: while order idempotents commute by Theorem 3.1, band projections need not commute. After this observation, it is not surprising that $BP(A)$ need not be a Boolean algebra: in the previous example, $BP(A)$ is not closed under taking suprema.

By contrast, the elements of $BP_l(A) \cap BP_r(A)$ always commute. Indeed, let $a, b \in BP_l(A) \cap BP_r(A)$. Since L_a, L_b, R_a, R_b are band projections, they commute, so $aba = L_a L_b(a) = L_b L_a(a) = ba^2$ and $aba = R_a R_b(a) = R_b R_a(a) = a^2 b$, which gives $ab = a^2 b = aba = ba^2 = ba$.

One could think that previous example is somewhat pathological, because $p^2 = 0$ and $L_p R_p = 0$. Maybe this is the only reason a band projection is not an order idempotent. It is not, and a counterexample is coming. But first an observation.

Remark 3.20. Let $\{A_i\}_{i \in I}$ be a family of Banach lattice algebras indexed on a set I . Consider $A = \ell_p(A_i)$ for some $1 \leq p \leq \infty$. We claim that

$$BP(\ell_p(A_i)) = \prod_{i \in I} BP(A_i).$$

Indeed, a positive element $(p_i)_{i \in I}$ is a band projection in $\ell_p(A_i)$ if and only if for all $(x_i)_{i \in I} \in \ell_p(A_i)_+$:

$$0 \leq (p_i)(x_i)(p_i) \leq (x_i) \quad \text{and} \quad (p_i)(x_i)(p_i) = (p_i)^2(x_i)(p_i)^2.$$

This is the same as saying that $0 \leq p_i x_i p_i \leq x_i$ and $p_i x_i p_i = p_i^2 x_i p_i^2$ whenever $x_i \in (A_i)_+$, for all $i \in I$. That is, $(p_i) \in BP(\ell_p(A_i))$ if and only if $p_i \in BP(A_i)$.

Similarly, one can check that

$$BP_l(\ell_p(A_i)) = \prod_{i \in I} BP_l(A_i) \quad \text{and} \quad BP_r(\ell_p(A_i)) = \prod_{i \in I} BP_r(A_i).$$

If each A_i has an identity, then also

$$OI(\ell_\infty(A_i)) = \prod_{i \in I} OI(A_i).$$

Example 3.21. Let A be the Banach lattice algebra of Example 3.18, and consider $\ell_\infty(A, A)$. By previous remark, the element

$$q = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

is a band projection that is not order idempotent. In this case, $q^2 \neq 0$ and $L_q R_q \neq 0$.

At this point, it is clear that the relation between order idempotents and band projections in Banach lattice algebras with identity is not trivial. The forthcoming results aim at a better understanding of it.

Proposition 3.22. *Let A be a Banach lattice algebra with identity e , and let $p \in BP(A)$. Then the following are equivalent:*

- (i) $p \in OI(A)$,
- (ii) p is idempotent,
- (iii) $p \in A_e$,
- (iv) $(\lambda e + p)^{-1} \geq 0$ for some $\lambda > \|p\|$.

Proof. Condition (i) clearly implies (ii). If p is idempotent, $L_p R_p(e) = p^2 = p$, and since $0 \leq L_p R_p(e) \leq e$, we have $p \in A_e$. So (ii) implies (iii). If $p \in A_e$, then, using that A_e is a subalgebra and sublattice of A , we can consider the restriction $L_p R_p: A_e \rightarrow A_e$. This operator is a band projection on A_e . Represent A_e as a $C(K)$, for a certain compact Hausdorff space K . Since $BP(C(K)) = OI(C(K))$, it follows that $p \in OI(A_e)$. But the order and product in A_e are the same as those in A , thus $p \in OI(A)$. This shows that (ii) implies (iii), and also representing A_e as $C(K)$ it is direct from $BP(C(K)) = OI(C(K))$ that (iii) implies (i).

It only remains to show the equivalence between (iii) and (iv). Since $p \in BP(A)$, we have $p^2 = L_p R_p(e) = (L_p R_p)^2(e) = p^4$. Inductively, $p^2 = p^{2n}$ and $p^3 = p^{2n+1}$ for all $n \geq 1$. Using this fact and the Neumann series, for every $\lambda > \max\{1, \|p\|\}$,

$$\begin{aligned} \lambda(\lambda e + p)^{-1} &= (e + p/\lambda)^{-1} = \sum_{n=0}^{\infty} \frac{(-p)^n}{\lambda^n} \\ &= e - \frac{p}{\lambda} + \frac{p^2}{\lambda^2} \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n}} - \frac{p^3}{\lambda^3} \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n}} \\ &= e - \frac{p}{\lambda} + \frac{p^2}{\lambda^2 - 1} - \frac{p^3}{\lambda(\lambda^2 - 1)} \\ &= \left(e - \frac{p}{\lambda}\right) \left(e + \frac{p^2}{\lambda^2 - 1}\right) \end{aligned}$$

Suppose that $p \in A_e$. Then $p \leq \|p\|_e e$ and, by the previous computation, $(\|p\|_e + 2)e + p$ is invertible with positive inverse. This shows that (iii) implies (iv). Conversely, if $(\lambda e + p)^{-1} \geq 0$ for some $\lambda > \|p\|$, either $p = 0$, in which case the result is trivial, or $p \neq 0$, in which case $\lambda > \|p\|_e \geq 1^\dagger$ and then

$$(\lambda e + p)(\lambda e - p) = \lambda^2 e - p^2 \geq 0.$$

Multiplying by $(\lambda e + p)^{-1}$ on the left

$$\lambda e - p = (\lambda e + p)^{-1}(\lambda^2 e - p^2) \geq 0$$

because the product of positive elements is positive. Hence $p \in A_e$. $\square$

In order to delve deeper into the relation between band projections and order idempotents, we need to introduce spectra in our study.

3.3 COMPLEXIFICATION AND SPECTRA

To talk about spectra and see their relation with band projections and order idempotents, we need to work over the complex field. So first we need to “complexify” some of the notions introduced so far.

Let A be a Banach lattice algebra. Applying the procedure for the complexification of Banach lattices, as exposed in Section 1.2.4, one obtains the complex Banach lattice $A_{\mathbb{C}}$ with norm $\|z\|_{\mathbb{C}} = \|z\|$. With the natural product

$$(x_1 + ix_2)(y_1 + iy_2) = (x_1 y_1 - x_2 y_2) + i(x_1 y_2 + x_2 y_1) \quad \text{where } x_1, x_2, y_1, y_2 \in A,$$

the space $A_{\mathbb{C}}$ becomes a complex algebra, that has an identity precisely when A does. This product is compatible with the complex lattice structure in the sense that $|z_1 z_2| \leq |z_1| |z_2|$, for all $z_1, z_2 \in A_{\mathbb{C}}$. This fact is not trivial, and a proof may be found in [Hui85]. The submultiplicativity of the norm on A

† Due to the fact that the spectral radius of p is 1, see Section 3.3.

immediately implies $\|z_1 z_2\|_{\mathbb{C}} \leq \|z_1\|_{\mathbb{C}} \|z_2\|_{\mathbb{C}}$ for $z_1, z_2 \in A_{\mathbb{C}}$. Hence $A_{\mathbb{C}}$ is also a complex Banach algebra with norm $\|\cdot\|_{\mathbb{C}}$. The space $A_{\mathbb{C}}$ with its structures of complex Banach lattice and complex Banach algebra is called a *complex Banach lattice algebra*. When A has an identity, we emphasize it by saying that $A_{\mathbb{C}}$ is a *complex Banach lattice algebra with identity*.

Example 3.23. (i) If K is a compact Hausdorff space, the complexification of $C(K)$ may be identified with $C(K, \mathbb{C})$, the space of continuous complex-valued functions on K .

(ii) If X is an order complete Banach lattice, the complexification of $\mathcal{L}_r(X)$ may be identified with the space of regular operators over $X_{\mathbb{C}}$ (see Section 1.2.4).

Next we want to present a result analogous to Theorem 3.1 in complex Banach lattice algebras. If X is a Banach lattice, a subspace $I \subseteq X_{\mathbb{C}}$ is called an *ideal* if $x \in I$ and $|y| \leq |x|$ imply $y \in I$. It is not difficult to check that I is the complexification of the ideal $I \cap X$ of X , and that the intersection of all the ideals containing $z \in X_{\mathbb{C}}$ is given by

$$(I_z)_{\mathbb{C}} = \{w \in X_{\mathbb{C}} : |w| \leq \lambda |z| \text{ for some } \lambda \in \mathbb{R}_+\}.$$

This is called the *ideal generated by z* .

An ideal $B \subseteq X_{\mathbb{C}}$ is called a *band* (resp. *projection band*) if $B \cap X$ is a band (resp. projection band) in X . Again, one can check that an ideal $B \subseteq X$ is a projection band if and only if $X_{\mathbb{C}} = B \oplus B^d$, where

$$B^d = \{z \in X_{\mathbb{C}} : |w| \wedge |z| = 0 \text{ for all } w \in B\}.$$

Finally, some common notation: for $z \in A_{\mathbb{C}}$ denote its spectrum by $\sigma(z)$ and its spectral radius by $r(z)$.

Theorem 3.24. *Let $A_{\mathbb{C}}$ be a complex Banach lattice algebra with identity e and let $(A_{\mathbb{C}})_e$ be the ideal generated by e in $A_{\mathbb{C}}$. Then $(A_{\mathbb{C}})_e = (A_e)_{\mathbb{C}}$, i.e., $(A_{\mathbb{C}})_e$ is the complexification of the Banach lattice algebra A_e . Moreover:*

(i) *The space $(A_{\mathbb{C}})_e$ can be isometrically identified with $C(K, \mathbb{C})$, for some compact Hausdorff K .*

(ii) *The ideal $(A_{\mathbb{C}})_e$ is a projection band in $A_{\mathbb{C}}$.*

(iii) *The space $(A_{\mathbb{C}})_e$ is an inverse closed subalgebra of $A_{\mathbb{C}}$.*

Proof. Since $(A_{\mathbb{C}})_e \cap A = A_e$, it follows that $(A_{\mathbb{C}})_e = (A_e)_{\mathbb{C}}$. It is then a direct consequence of Theorem 3.1 that $(A_{\mathbb{C}})_e$, with the complexification of the norm $\|\cdot\|_e$, is isometric to $C(K, \mathbb{C})$. To prove (i) it only remains to check that the norm $\|\cdot\|_{e, \mathbb{C}}$ is equal to $\|\cdot\|_{\mathbb{C}}$. Being A_e an inverse closed subalgebra, it is easy to check that the spectrum of an element in A_e (i.e., its range when viewed as a function in $C(K, \mathbb{C})$) is the same as its spectrum in $A_{\mathbb{C}}$. Therefore $\|z\| \geq r(z) =$

$\|z\|_e$, and since we are assuming $\|e\| = 1$ in our definition of Banach lattice algebras, $\|z\| \leq \|z\|_e$. Joining both inequalities, $\|z\| = \|z\|_e = r(z)$ and (i) is proved. From Theorem 3.1 clearly follows (ii), and exactly the same proof as in Theorem 3.1 can be used to show (iii). $\square$

The previous result completes the generalization from Theorem 1.8 to Theorem 3.1. It also completes Example 3.9. Some remarks regarding the spectrum of order idempotents and band projections are in order.

Remark 3.25. (i) Let p be an order idempotent. When we represent $(A_{\mathbb{C}})_e$ as a space of continuous functions, p is the characteristic function of a clopen set. As such, $\sigma(p) = \{0, 1\}$ whenever $p \neq 0, e$. By previous theorem, $\|p\| = r(p) = 1$.

(ii) If $p \in BP(A)$, either $p^2 = 0$, in which case $r(p) = 0$ and $\sigma(p) = \{0\}$ (the complexification of Example 3.18 serves as an example of this), or $p^2 \neq 0$, in which case $\|p\| = r(p) = \lim_n \|p^{2n}\|^{1/2n} = 1$ (because p^2 is order idempotent, so $\|p^2\| = 1$, and also $(p^2)^n = p^2$). In both cases, $\sigma(p^2) \subseteq \{0, 1\}$ implies $\sigma(p) \subseteq \{-1, 0, 1\}$.

(iii) For $p \in BP(A)$ to be an order idempotent, it is necessary that $\sigma(p) = \{0, 1\}$. However, it is not sufficient: one can easily check that the element q in Example 3.21 satisfies $\sigma(q) = \{0, 1\}$, yet it is not an order idempotent.

The following is an example of a band projection with spectrum $\{0, 1, -1\}$.

Example 3.26. In the Banach lattice algebra $M_3(\mathbb{R})$ of 3×3 real matrices (with entry-wise order and the usual matrix product, as in Example 3.18), consider the 3-dimensional subspace:

$$\begin{aligned} A &= \text{span} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. \end{aligned}$$

Being generated by pairwise disjoint elements, A is a sublattice. Moreover, the identities

$$\begin{aligned} \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix} \begin{pmatrix} x' & 0 & y' \\ 0 & z' & 0 \\ y' & 0 & x' \end{pmatrix} &= \begin{pmatrix} xx' + yy' & 0 & xy' + x'y \\ 0 & zz' & 0 \\ xy' + x'y & 0 & xx' + yy' \end{pmatrix}, \\ \begin{pmatrix} x & 0 & y \\ 0 & z & 0 \\ y & 0 & x \end{pmatrix}^{-1} &= \frac{1}{x^2 - y^2} \begin{pmatrix} x & 0 & -y \\ 0 & z^{-1}(x^2 - y^2) & 0 \\ -y & 0 & x \end{pmatrix}, \end{aligned}$$

when both $x^2 \neq y^2$ and $z \neq 0$, prove that A is an inverse closed subalgebra, with identity because the identity matrix is contained in A . Therefore, A is a Banach

lattice algebra with the inherited product, order and norm. Its complexification consists in taking matrices with complex entries.

Set

$$p = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is not difficult to check that $pap = a$ for every $a \in A_{\mathbb{C}}$.^{††} Therefore $L_p R_p = \text{id}_{A_{\mathbb{C}}}$, and this implies $p \in BP(A)$. The spectrum of p is the set of roots of its characteristic polynomial

$$\det(p - \lambda \text{id}_{\mathbb{R}^3}) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda + 1)(\lambda - 1),$$

which is $\sigma_A(p) = \sigma(p) = \{-1, 0, 1\}$.

Next we want to provide a result that relates spectra with order idempotents. We start with a simple observation.

Remark 3.27. In $C(K)$ order idempotents are characterized by their spectrum. More precisely, $p \in OI(C(K))$ if and only if $\sigma(p) \subseteq \{0, 1\}$.

We also need the generalization of Theorem 1.10 to arbitrary complex Banach lattice algebras with identity.

Corollary 3.28. *Let $A_{\mathbb{C}}$ be a complex Banach lattice algebra with identity. Let $a \in A_+$ be an invertible element with positive inverse. Then $a \in A_e$ if and only if $\sigma(a) \subseteq \mathbb{R}_+$.*

Proof. Let $a \in A_+$ be an invertible element such that $a^{-1} \in A_+$. Then $R_a: A \rightarrow A$, being a positive operator with positive inverse $R_{a^{-1}}$, is a lattice isomorphism. Note that

$$(R_a - \lambda \text{id}_{A_{\mathbb{C}}})(x) = x(a - \lambda e) = R_{a - \lambda e}(x) \quad \text{for all } x \in A_{\mathbb{C}},$$

where e denotes the identity element of $A_{\mathbb{C}}$. It follows that $\sigma(R_a) \subseteq \sigma(a)$, since $R_{a - \lambda e}$ is invertible whenever $a - \lambda e$ is. Suppose $\sigma(a) \subseteq \mathbb{R}_+$. Then $\sigma(R_a) \subseteq \mathbb{R}_+$ and R_a is a lattice isomorphism. By Theorem 1.10, $R_a \in \mathcal{Z}(A)$, which readily implies $a \in A_e$. The converse is direct if we see $a \in A_e$ as a positive continuous function on a compact space. $\square$

Proposition 3.29. *Let $A_{\mathbb{C}}$ be a complex Banach lattice algebra with identity. Let $a \in A_+$ be an invertible element such that:*

$$(i) \quad a^{-1} \in A_+,$$

^{††}Using elementary matrix theory: right multiplication by p swaps the first and third columns of $a \in A_{\mathbb{C}}$, while left multiplication swaps the first and third rows of a ; that is, pap is the reflection of a with respect to both of its diagonals. The elements of A , however, are invariant under this operation.

(ii) $\sigma(a) \subseteq \{\lambda, \lambda + 1\}$ for some $\lambda \in \mathbb{R}_+$.

Then $a - \lambda e \in OI(A)$, where e is the identity of A .

Proof. Since a has positive spectrum, and $a^{-1} \in A_+$, from previous corollary it follows that $a \in A_e$. Then $a - \lambda e \in A_e$, and its spectrum satisfies, by hypothesis, $\sigma(a - \lambda e) \subseteq \{0, 1\}$. Since A_e is essentially a space of continuous functions, it follows that $a - \lambda e$ is an order idempotent. $\square$

3.4 INNER BAND PROJECTIONS

In this section the construction of inner band projections explained in Section 2.1.1 is extended to general Banach lattice algebras. Let A be an order complete Banach lattice algebra and let $\{p_\lambda\}_{\lambda \in \Lambda} \subseteq BP_l(A) \cap BP_r(A)$ be such that $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$ for all $\alpha, \beta \in \Lambda$ (in particular, if A has an identity, we can take the p_λ to be pairwise disjoint order idempotents). For $\Gamma \subseteq \Lambda \times \Lambda$ define the map

$$\begin{aligned} P_\Gamma: A_+ &\longrightarrow A_+ \\ x &\longmapsto \bigvee_{(\alpha, \beta) \in \Gamma} p_\alpha x p_\beta. \end{aligned} \quad (3.2)$$

Note that this is well-defined, because $p_\alpha x p_\beta = L_{p_\alpha} R_{p_\beta}(x) \leq x$ (recall that $L_{p_\alpha} R_{p_\beta}$ is a band projection on A), and A is order complete. If we do not assume that A is order complete, the coming results work just fine, as long as we ask for this supremum to exist. In particular, if Γ is finite, everything works out.

Lemma 3.30. *Let A be a Banach lattice algebra, and let $(p_\lambda)_{\lambda \in \Lambda} \subseteq BP_l(A) \cap BP_r(A)$ be such that $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$. If $\Phi \subseteq \Lambda \times \Lambda$ is finite, then*

$$\bigvee_{(\alpha, \beta) \in \Phi} p_\alpha x p_\beta = \sum_{(\alpha, \beta) \in \Phi} p_\alpha x p_\beta, \quad \text{for all } x \in A_+.$$

Proof. For every $(\alpha, \beta) \in \Phi$, $L_{p_\alpha} R_{p_\beta}$ is a band projection; to ease the notation, denote it by $L_\alpha R_\beta$. Whenever $(\gamma, \delta) \neq (\alpha, \beta)$:

$$(L_\alpha R_\beta)(L_\gamma R_\delta)(x) = p_\alpha p_\gamma x p_\delta p_\beta = 0$$

because either $p_\alpha p_\gamma = 0$ or $p_\delta p_\beta = 0$. Therefore $(L_\alpha R_\beta)(L_\gamma R_\delta) = 0$. Being band projections, this implies that the bands determined by $L_\alpha R_\beta$ and $L_\gamma R_\delta$ are disjoint. Thus

$$p_\alpha x p_\beta \wedge p_\gamma x p_\delta = (L_\alpha R_\beta)(x) \wedge (L_\gamma R_\delta)(x) = 0,$$

or, equivalently, $p_\alpha x p_\beta \vee p_\gamma x p_\delta = p_\alpha x p_\beta + p_\gamma x p_\delta$. Since this is true for any pair of indices, the result follows by induction. $\square$

Theorem 3.31. *The map P_Γ , as defined in (3.2), extends to a unique band projection on A .*

Proof. Note first that, using the previous lemma,

$$P_\Gamma x = \bigvee_{(\alpha,\beta) \in \Gamma} p_\alpha x p_\beta = \sup_{\Phi \in \text{Fin}(\Gamma)} \bigvee_{(\alpha,\beta) \in \Phi} p_\alpha x p_\beta = \sup_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha,\beta) \in \Phi} p_\alpha x p_\beta. \quad (3.3)$$

It follows from the definition that $P_\Gamma(x + y) \leq P_\Gamma(x) + P_\Gamma(y)$. For every $\Phi, \Psi \in \text{Fin}(\Gamma)$:

$$\begin{aligned} \sum_{(\alpha,\beta) \in \Phi} p_\alpha x p_\beta + \sum_{(\gamma,\delta) \in \Psi} p_\gamma y p_\delta &\leq \sum_{(\alpha,\beta) \in \Phi \cup \Psi} p_\alpha x p_\beta + \sum_{(\gamma,\delta) \in \Phi \cup \Psi} p_\gamma y p_\delta \\ &= \sum_{(\alpha,\beta) \in \Phi \cup \Psi} p_\alpha (x + y) p_\beta \leq P_\Gamma(x + y). \end{aligned}$$

Taking suprema separately in Φ and Ψ we conclude that P_Γ is additive. Hence it extends to a unique positive operator $P_\Gamma: A \rightarrow A$. It is clear from the definition that $0 \leq P_\Gamma \leq \text{id}_A$. To show that it is a band projection, it only remains to prove that it is idempotent. By previous inequality, $P_\Gamma^2 \leq P_\Gamma$. To see the other inequality, note that for an arbitrary $\Phi \in \text{Fin}(\Gamma)$:

$$\begin{aligned} P_\Gamma(P_\Gamma(x)) &= \sum_{(\alpha,\beta) \in \Phi} p_\alpha P_\Gamma(x) p_\beta \geq \sum_{(\alpha,\beta) \in \Phi} p_\alpha \left(\sum_{(\gamma,\delta) \in \Phi} p_\gamma x p_\delta \right) p_\beta \\ &= \sum_{(\alpha,\beta) \in \Phi} \sum_{(\gamma,\delta) \in \Phi} p_\alpha p_\gamma x p_\delta p_\beta \\ &= \sum_{(\alpha,\beta) \in \Phi} p_\alpha x p_\beta. \end{aligned}$$

Inequality $P_\Gamma^2 \geq P_\Gamma$ follows from taking supremum on $\Phi \in \text{Fin}(\Gamma)$. $\square$

Definition 3.32. Let A be a Banach lattice algebra and let $P: A \rightarrow A$ be a band projection. If there exist $(p_\lambda)_{\lambda \in \Lambda} \subseteq BP_l(A) \cap BP_r(A)$ such that $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$ for all $\alpha, \beta \in \Lambda$, and such that $P = P_\Gamma$ for some $\Gamma \subseteq \Lambda \times \Lambda$ as defined in (3.2), then we say that P is an *inner band projection*. We say that the band $P(A)$ is an *inner projection band*.

Example 3.33. Let K be a compact Hausdorff space. Even though $C(K)$ may not be order complete, it still makes sense to consider inner band projections, as long as the supremum in (3.2) is well-defined. In fact, in $C(K)$, any band projection is inner, because it consists in multiplying by the characteristic function of a clopen set, which is an order idempotent.

Remark 3.34. Let A be an order complete Banach lattice algebra. Since in $\mathcal{L}_r(A)$ the supremum of an increasing net is taken pointwise, equation (3.3) implies

$$P_\Gamma = \text{o-lim}_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha,\beta) \in \Phi} L_\alpha R_\beta,$$

where o-lim denotes the limit of the net in the order convergence. In particular, if A is order continuous, we have

$$P_\Gamma(x) = \lim_{\Phi \in \text{Fin}(\Gamma)} \sum_{(\alpha, \beta) \in \Phi} p_\alpha x p_\beta = \sum_{(\alpha, \beta) \in \Gamma} p_\alpha x p_\beta.$$

Proposition 3.35. *Let A be an order complete Banach lattice algebra, and let $(p_\lambda)_{\lambda \in \Lambda} \subseteq BP_l(A) \cap BP_r(A)$ be such that $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$. The set $\{P_\Gamma : \Gamma \subseteq \Lambda \times \Lambda\}$ forms a Boolean algebra with maximum element $P_{\Lambda \times \Lambda}$, minimum $P_\emptyset$ and operations*

$$P_\Gamma \wedge P_\Delta = P_{\Gamma \cap \Delta}, \quad P_\Gamma \vee P_\Delta = P_{\Gamma \cup \Delta}, \quad \overline{P_\Gamma} = P_{\overline{\Gamma}}, \quad \text{for } \Gamma, \Delta \subseteq \Lambda \times \Lambda,$$

where $\overline{\Gamma} = \Lambda \times \Lambda \setminus \Gamma$ denotes the complementary set.

Proof. On the set of inner band projections, consider the usual order of operators. Let $\Gamma, \Delta \subseteq \Lambda \times \Lambda$. We first want to prove that $P_\Gamma \wedge P_\Delta = P_{\Gamma \cap \Delta}$. For every $x \in A_+$, the net $(\sum_{(\gamma, \delta) \in \Psi} p_\gamma x p_\delta)_{\Psi \in \text{Fin}(\Delta)}$ increases to $P_\Delta(x)$. For every $(\alpha, \beta) \in \Gamma$, using that band projections are order continuous:

$$L_{p_\alpha} R_{p_\beta}(P_\Delta(x)) = \text{o-lim}_{\Psi \in \text{Fin}(\Delta)} \sum_{(\gamma, \delta) \in \Psi} L_{p_\alpha} R_{p_\beta}(p_\gamma x p_\delta) = \begin{cases} p_\alpha x p_\beta & \text{if } (\alpha, \beta) \in \Delta, \\ 0 & \text{if } (\alpha, \beta) \notin \Delta. \end{cases}$$

Therefore

$$P_\Gamma(P_\Delta(x)) = \bigvee_{(\alpha, \beta) \in \Gamma} L_{p_\alpha} R_{p_\beta}(P_\Delta(x)) = \bigvee_{(\alpha, \beta) \in \Gamma \cap \Delta} p_\alpha x p_\beta = P_{\Gamma \cap \Delta}(x),$$

as wanted.

To check the formula for the supremum, first consider the situation $\Gamma \cap \Delta = \emptyset$. In this case, $P_\Gamma P_\Delta = 0$, and

$$P_\Gamma + P_\Delta = P_\Gamma \vee P_\Delta \leq P_{\Gamma \cup \Delta}$$

because both $P_\Gamma \leq P_{\Gamma \cup \Delta}$ and $P_\Delta \leq P_{\Gamma \cup \Delta}$. To see the reverse inequality, take finite sets $\Phi \in \text{Fin}(\Gamma)$ and $\Psi \in \text{Fin}(\Delta)$. Using that they are disjoint:

$$P_\Gamma(x) + P_\Delta(x) \geq \sum_{(\alpha, \beta) \in \Phi} p_\alpha x p_\beta + \sum_{(\gamma, \delta) \in \Psi} p_\gamma x p_\delta = \sum_{(\alpha, \beta) \in \Phi \cup \Psi} p_\alpha x p_\beta.$$

The right hand side is an arbitrary finite subset of $\Gamma \cup \Delta$; taking supremum over these yields $P_\Gamma + P_\Delta \geq P_{\Gamma \cup \Delta}$. In particular, when $\Delta = \overline{\Gamma}$, we get the desired formula for the complementary. Finally, for arbitrary Γ and Δ (not necessarily disjoint), write their union as the three disjoint unions $\Gamma \cup \Delta = (\overline{\Gamma} \cap \Delta) \cup (\Gamma \cap \overline{\Delta}) \cup (\Gamma \cap \Delta)$ and apply the previous formulae. $\square$

Remark 3.36. When $A = \mathcal{L}_r(X)$, for some order complete Banach lattice X , the Boolean algebra of inner band projections is isomorphic to $2^{\Lambda \times \Lambda}$, according to Proposition 2.10. In the general case, however, this need not be true. In Example 3.16, the only inner band projection is the zero projection, even though there are many pairwise disjoint left and right band projections.

We finish with an illustration of inner band projections in a Banach lattice algebra without identity. This example also shows that an analogue of Theorem 2.19 for this general setup is not straightforward.

Example 3.37. Consider the 3-dimensional Banach lattice algebra

$$\begin{aligned} A &= \text{span} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq M_3(\mathbb{R}), \end{aligned}$$

where the order, product and norm are inherited from $M_3(\mathbb{R})$ (with the conventions as in Example 3.18). It is not difficult to check that this Banach lattice algebra does not have an identity. By direct computation, one can also check that

$$\begin{aligned} BP_l(A) \cap BP_r(A) &= \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, p_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

Since $p_1 a p_2 = 0 = p_2 a p_1$ for $a \in A$, the only inner band projections besides the zero projection are $a \mapsto p_1 a p_1$, $a \mapsto p_2 a p_2$, and their sum; explicitly, the only inner band projections are

$$\begin{aligned} \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ & \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to check that

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}$$

defines a band projection that is not inner.

This chapter has translated much of Chapter 2 into the abstract setting: order idempotents play the role of band projections on the underlying lattice, the ideal A_e plays the role of the center, and P_Γ generalizes the inner band projections $\mathcal{P}_\Gamma$. What the abstraction leaves open is whether any of it is *actually needed*—whether every Banach lattice algebra already embeds faithfully, as a sublattice-algebra, into $\mathcal{L}_r(X)$ for some order complete Banach lattice X . We turn to this question in Chapter 4.

4 WICKSTEAD'S CONJECTURE

David Muñoz-Lahoz, *Wickstead's conjecture on positive projections and non-representable banach lattice algebras*, Preprint, arXiv:2604.14697, 2026

The two preceding chapters took $\mathcal{L}_r(X)$, for X an order complete Banach lattice, as the guiding example of a Banach lattice algebra, and isolated which of its lattice and algebraic features—inner band projections, the ideal generated by the identity, the Boolean algebra of order idempotents, the interplay between order idempotents and band projections—can be formulated and proved in an arbitrary order complete Banach lattice algebra. This line of work leaves open the converse question: does that abstraction introduce genuinely new objects, or is every Banach lattice algebra already a sublattice-algebra of some $\mathcal{L}_r(X)$? Put differently, does every Banach lattice algebra A admit a faithful lattice and algebra homomorphism into $\mathcal{L}_r(X)$, for some order complete vector lattice X ? This is the representation problem for Banach lattice algebras, flagged in the introduction as one of the central open problems in the area and posed explicitly by Wickstead as [Wic17a, Question 5.3] and [Wic17b, Problem 3].

The interest in a positive answer parallels the role played by concrete representation theorems in operator algebras, where abstractly defined structures are shown to embed faithfully into bounded operators on a Hilbert space. The paradigmatic instance is the Gelfand–Naimark–Segal theorem, which represents any C^* -algebra as a closed $*$ -subalgebra of bounded operators on some Hilbert space. A faithful representation theorem for Banach lattice algebras would offer the analogous structural cornerstone, reducing the study of an abstract Banach lattice algebra to the concrete, well-understood machinery of regular operators on order complete Banach lattices.

The main goal of this chapter is to settle the representation problem *in the negative*. We exhibit Banach lattice algebras with identity of dimension 2 that admit no faithful representation on any order complete vector lattice (Example 4.22).

The obstruction we exploit passes through a question on positive projections, also due to Wickstead. Suppose $\Phi: A \rightarrow \mathcal{L}_r(X)$ is a faithful representation of a Banach lattice algebra A , with X order complete, and suppose A contains two non-zero positive idempotents e and p together with a scalar $\alpha \geq 0$ for which $ep = pe = p$ and $p - \alpha e$ is disjoint from e . Setting $Y = \Phi(e)(X)$ —itself an order complete vector lattice when endowed with the order inherited from X —the operator $\Phi(p)$ restricts to a positive projection $P: Y \rightarrow Y$ satisfying

- (i) $\alpha \text{id}_Y \leq P$, and

- (ii) whenever $T \in \mathcal{L}_r(Y)$ is a positive operator such that $T \leq \lambda \text{id}_Y$ for some $\lambda \geq 0$ and $T \leq P$, then also $T \leq \alpha \text{id}_Y$.

Thus a faithful representation of A forces the scalar α to appear as a parameter of a concrete positive projection on an order complete vector lattice satisfying (i) and (ii). Ruling out certain values of α here accordingly rules out the corresponding Banach lattice algebras from being representable.

Precisely this restriction on α was conjectured by Wickstead in [Wic17c]. When X is finite-dimensional, conditions (i) and (ii) force the diagonal entries of the matrix representing P to be all equal to α . We shall accordingly say that such a P has *constant diagonal* α whenever the two conditions are satisfied. Wickstead conjectured that the only values of α compatible with (i) and (ii) are $\alpha = 0$ and $\alpha = 1/n$, for $n \in \mathbb{N}$, and he proved it when X is finite-dimensional. The infinite-dimensional case was left open. In Corollary 4.15 we show that the conjecture holds on every order complete vector lattice. The non-representable examples announced above then fall out at once: Wickstead listed in [Wic17c, Section 7] an explicit family of Banach lattice algebras on $\mathbb{R}^2$ that take values of α outside $\{0\} \cup \{1/n : n \in \mathbb{N}\}$.

The positive-projections result is itself obtained from a sharper theorem for $C(K)$ spaces (Theorem 4.3). On the other hand, Proposition 4.16 extends Wickstead's explicit description of contractive positive projections with constant diagonal on finite-dimensional Banach lattices to ℓ_p -spaces over arbitrary index sets, for $1 \leq p < \infty$.

4.1 PRELIMINARY DEFINITIONS AND NOTATION

Before diving into the material, let us review some standard notational conventions that will appear intensively in this chapter. For a compact Hausdorff space K , we denote as usual the space of continuous functions on K by $C(K)$ and its dual, which we identify with the space of all finite signed regular Borel measures on K , by $M(K)$. For every $f \in C(K)$ and $\mu \in M(K)$, we denote the dual pairing by $\langle \mu, f \rangle$, whilst the value of μ at a Borel set B will be denoted by $\mu(B)$. The point mass measure at $t \in K$ will be denoted by δ_t . Every $f \in C(K)$ has an associated multiplication operator $M_f: C(K) \rightarrow C(K)$ defined by $M_f(g) = fg$, for all $g \in C(K)$.

It is also convenient to introduce a couple of definitions. Let X be an order complete vector lattice. Recall from Section 1.2.3 that the diagonal projection $\mathcal{D}: \mathcal{L}_r(X) \rightarrow \mathcal{L}_r(X)$ is the projection onto the center of X .

Definition 4.1. Let X be an order complete vector lattice, and let $T: X \rightarrow X$ be a regular operator. We say that T has *constant diagonal* α if $\mathcal{D}(T) = \alpha \text{id}_X$.

Notice that T having constant diagonal α is the same as αid_X being the supremum of all the central operators that lie below T . This is the characterization we used at the beginning of the chapter.

Definition 4.2. A *representation* of a lattice-ordered algebra A is a lattice and algebra homomorphism $\Phi: A \rightarrow \mathcal{L}_r(X)$, for some order complete vector lattice X . We say that Φ is *faithful* if it is injective. In the case that A has an algebraic identity e and $\Phi(e) = \text{id}_X$, we say that Φ is *unital*.

These and other kinds of representations will be studied in more detail in Chapter 5.

4.2 THE MAIN THEOREM

This section is devoted to proving the following.

Theorem 4.3. *Let K be a nonempty compact Hausdorff space. Let $P: C(K) \rightarrow C(K)$ be a positive projection with $P\mathbf{1}_K = \mathbf{1}_K$. Suppose there exists some $\alpha \geq 0$ satisfying:*

- (i) $\alpha \text{id}_{C(K)} \leq P$, and
- (ii) if $0 \leq M_f \leq P$, then $\|f\|_\infty \leq \alpha$.

Then either $\alpha = 0$ or $\alpha = 1/n$ for some $n \in \mathbb{N}$.

We will construct the proof incrementally along this section. We start with some lemmas that may be of interest by themselves.

Lemma 4.4. *Let X be a Banach lattice, and let $P: X \rightarrow X$ be a positive projection. Let $S \subseteq X_+$ be a set, and let I be the ideal generated by S . If $P(S) \subseteq I$, then also $P(I) \subseteq I$ and $P(\bar{I}) \subseteq \bar{I}$.*

Proof. If $x \in I$, then there exist $s_1, \dots, s_n \in S$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ such that

$$|x| \leq \lambda_1 s_1 + \dots + \lambda_n s_n.$$

Using the positivity of P it follows that

$$|Px| \leq P|x| \leq \lambda_1 P s_1 + \dots + \lambda_n P s_n.$$

Since $P(S) \subseteq I$, the right hand side in the previous equation belongs to I , and therefore $Px \in I$. Thus $P(I) \subseteq I$ and, since P is continuous, $P(\bar{I}) \subseteq \bar{I}$. $\square$

Lemma 4.5. *Let X be a Banach lattice with a quasi-interior point. Let $P: X \rightarrow X$ be a positive projection satisfying $\alpha \text{id}_X \leq P$, for some $\alpha > 0$. If $B \subseteq X$ is a projection band invariant under P , then so is $B^\perp$.*

Proof. Let e be a quasi-interior point of X , and let $f = Pe$. Since $f = Pe \geq \alpha e$, and $\alpha > 0$, f is also a quasi-interior point. Let $Q: X \rightarrow X$ be the band projection onto B . Then using the positivity of P , and the fact that $Qf \leq f$, it follows that

$$PQf \leq Pf = f.$$

Since $PQf \in B$, applying Q to this equality yields $PQf \leq Qf$. Thus

$$0 = P(Qf - PQf) \geq \alpha(Qf - PQf) \geq 0$$

and the equality $PQf = Qf$ follows. Using this, the computation

$$(\text{id}_X - Q)f = f - Qf = Pf - PQf = P(\text{id}_X - Q)f$$

shows that $(\text{id}_X - Q)f$ belongs to the image of P . By Lemma 4.4, the closed ideal generated by $(\text{id}_X - Q)f$ is invariant under P . Since f is a quasi-interior point, this closed ideal is precisely B^d . Indeed, for every $x \in X$, find a sequence (x_n) contained in the ideal generated by f and such that $x_n \rightarrow x$ (this we can do because f is a quasi-interior point). Then $(\text{id}_X - Q)x_n \rightarrow (\text{id}_X - Q)x$, where the sequence $((\text{id}_X - Q)x_n)$ belongs to the ideal generated by $(\text{id}_X - Q)f$. $\square$

Recall that a projection $P: X \rightarrow X$ on a vector lattice X is said to be *strictly positive* if $x > 0$ implies $Px > 0$.

Lemma 4.6. *Let X be a Banach lattice, and let $P: X \rightarrow X$ be a strictly positive projection. If the only closed ideals invariant under P are the trivial ones (that is, $\{0\}$ and X), then P has rank 1.*

Proof. Since P is strictly positive, by [AA02a, Lemma 5.43] $P(X)$ is a (closed) vector sublattice of X . If $P(X)$ didn't have dimension 1, then there would exist two positive disjoint vectors $x, y \in P(X)$. The closed ideal generated by x in X is non-trivial (because y does not belong to the band generated by x), and yet invariant by Lemma 4.4. This contradicts the assumption. $\square$

Let K be a compact Hausdorff space, and let $P: C(K) \rightarrow C(K)$ and $\alpha \geq 0$ be as in Theorem 4.3. If $\alpha = 0$ we are done, so we may assume from now on that $\alpha > 0$. The adjoint $P^*: M(K) \rightarrow M(K)$ is again a positive projection satisfying $\alpha \text{id}_{M(K)} \leq P^*$. For every $t \in K$, let $\mu_t = P^*\delta_t$. Notice that the map

$$\begin{array}{ccc} K & \longrightarrow & M(K) \\ t & \longmapsto & \mu_t \end{array}$$

is continuous for the weak* topology on $M(K)$; this fact will be used several times later on.

Decompose

$$\mu_t = \sum_{s \in J_t} \lambda(t, s) \delta_s + \nu_t,$$

where ν_t is a positive continuous measure (i.e., it is disjoint from every point mass measure), $J_t \subseteq K$ is an at most countable set of points, and $\lambda(t, s) > 0$ for every $s \in J_t$. Moreover, since $\alpha \text{id}_{M(K)} \leq P^*$ and $\alpha > 0$, we must have $t \in J_t$ and $\alpha \leq \lambda(t, t)$ for all $t \in K$. Also, it is important to keep in mind that μ_s is positive and

$$\|\mu_s\| = \mu_s(K) = \langle P^*\delta_s, \mathbb{1}_K \rangle = \langle \delta_s, P\mathbb{1}_K \rangle = \langle \delta_s, \mathbb{1}_K \rangle = 1.$$

Lemma 4.7. *For every $s \in J_t$, $J_s \subseteq J_t$.*

Proof. Using that P^* is a projection, we have

$$\sum_{s \in J_t} \lambda(t, s) \delta_s + \nu_t = \mu_t = P^* \mu_t = \sum_{s \in J_t} \lambda(t, s) \mu_s + P^* \nu_t. \quad (4.1)$$

Notice that all the coefficients are positive. Hence, for this equality to hold, the discrete part of each μ_s cannot contain any point mass measure supported outside J_t . $\square$

Fix an arbitrary $t \in K$. Denote by I the order ideal generated by $\{\mu_s : s \in J_t\}$ in $M(K)$. Since $P^* \mu_s = \mu_s$ for every $s \in J_t$, Lemma 4.4 guarantees that both I and $\bar{I}$ are invariant under P^* . Since $M(K)$ is order continuous, $\bar{I}$ is actually a projection band.

Lemma 4.8. *The only closed invariant ideals of $P^*|_{\bar{I}}$ are the trivial ones.*

Proof. We want to use Lemma 4.5. For this, first note that $\bar{I}$, being a band in $M(K)$, is itself an order continuous Banach lattice (actually, it is an AL-space). Also note that $\bar{I}$ has a quasi-interior point: if $J_t = \{t_i\}_{i=1}^\infty$ is an enumeration of J_t , then $e = \sum_{i=1}^\infty 2^{-i} \mu_{t_i}$ is a quasi-interior point of $\bar{I}$ (recall that $\|\mu_s\| = 1$).

Suppose $I' \subseteq \bar{I}$ is a closed ideal invariant under $P^*|_{\bar{I}}$. The order continuity of $\bar{I}$ implies that I' is a projection band in $\bar{I}$. By Lemma 4.5, its disjoint complement in $\bar{I}$ is also invariant under $P^*|_{\bar{I}}$. Now $\delta_t \in \bar{I}$ is an atom, and therefore must belong to either I' or its disjoint complement. Without loss of generality, assume it belongs to I' . Then the invariance of I' under P^* implies that

$$P^* \delta_t = \mu_t = \sum_{s \in J_t} \lambda(t, s) \delta_s + \nu_t \in I'.$$

Since $\lambda(t, s) > 0$ for every $s \in J_t$, and I' is an ideal, $\delta_s \in I'$ and, using again invariance, $\mu_s \in I'$ for all $s \in J_t$. But $\bar{I}$ is precisely the closed ideal generated by μ_s , so that $I' = \bar{I}$. $\square$

By Lemma 4.6, $P^*(\bar{I})$ has rank one. That is, there exist a measure $\rho \in M(K)_+$ and scalars $\lambda_s \in \mathbb{R}_+$ such that $\mu_s = P^* \mu_s = \lambda_s \rho$ for all $s \in J_t$. But since $\mu_s(K) = 1$ for every $s \in J_t$, λ_s does not depend on s , and therefore $\mu_s = \mu_t$ for every $s \in J_t$. This implies that the family of all distinct sets J_t forms a partition of K .

Lemma 4.9. *For all $s \in K$, either $J_s = J_t$ (when $s \in J_t$) or $J_s \cap J_t = \emptyset$ (when $s \notin J_t$).*

Proof. When $s \in J_t$, we have noted already that $\mu_s = \mu_t$, so obviously $J_s = J_t$. If $s' \in J_s \cap J_t$, then again we would have $\mu_s = \mu_{s'} = \mu_t$, and therefore $s \in J_s = J_t$. $\square$

The fact that $\mu_s = \mu_t$ for all $s \in J_t$ has further implications on the expression of μ_t . First, since the continuous parts of μ_t and μ_s must be equal, $\nu_t = \nu_s$ holds for every $s \in J_t$. Second, since the coefficient of each point evaluation must be equal, $\lambda(t, s) = \lambda(s, s)$ holds for every $s \in J_t$. Hence $\lambda(t, s) \geq \alpha$ for every $s \in J_t$. Moreover, since $\alpha > 0$ and $\|\mu_s\| = 1$, J_t must be finite and satisfy $|J_t| \leq 1/\alpha$.

Next we want to show that $\nu_t = 0$. For this we need the following topological lemma.

Lemma 4.10. *For every $\varepsilon > 0$ and $t' \in J_t$, there exists an open neighborhood V of t' such that $(P^*\nu_t)(V) < \varepsilon$.*

Proof. Let $J_t = \{t_1, \dots, t_n\}$, and let $U'_1, \dots, U'_n$ be pairwise disjoint open neighborhoods of $t_1, \dots, t_n$, respectively. Let $U' = U'_1 \cup \dots \cup U'_n$. Since ν_t is a regular continuous measure, we may choose these neighborhoods small enough so as to have $\nu_t(U') < \varepsilon/2$. Let also U_i be an open neighborhood of t_i satisfying $\overline{U_i} \subseteq U'_i$ for $i = 1, \dots, n$, and denote $U = U_1 \cup \dots \cup U_n$.

Fix $t' \in J_t$. For every $s \notin U$, J_s does not contain any of the points in J_t by Lemma 4.9. Therefore there exists an open neighborhood V_s of t' such that $\mu_s(V_s) < \varepsilon/2$. Recall that the map $s' \mapsto \mu_{s'}$ is continuous in the weak* topology. Hence there exists an open neighborhood W_s of s such that $\mu_{s'}(V_s) < \varepsilon/2$ for all $s' \in W_s$.

This way the $\{W_s\}_{s \in K \setminus U}$ form an open covering of the set $K \setminus U$. Since the latter is compact, we can extract a finite subcovering:

$$K \setminus U \subseteq W_{s_1} \cup \dots \cup W_{s_m}.$$

Consider the set $V = V_{s_1} \cap \dots \cap V_{s_m}$, which is again an open neighborhood of t' . Notice that, if $s \notin U$, then $s \in W_{s_i}$ for some $i \in \{1, \dots, m\}$, and therefore

$$\mu_s(V) \leq \mu_s(V_{s_i}) < \varepsilon/2.$$

To prove that V is the set we wanted to construct, we need to show that, for every $f \in C(K)$ satisfying $0 \leq f \leq \mathbb{1}_K$ and $\text{supp } f \subseteq V$, we have

$$\langle P^*\nu_t, f \rangle = \langle \nu_t, Pf \rangle < \varepsilon.$$

Fix such an $f \in C(K)$. Let $U_1 = U'$, $U_2 = K \setminus \overline{U}$, and let $\{f_1, f_2\}$ be a partition of unity subordinate to the open covering $\{U_1, U_2\}$. On the one hand, for every $s \notin U$:

$$(Pf)(s)f_2(s) \leq \langle P^*\delta_s, f \rangle = \langle \mu_s, f \rangle \leq \mu_s(V) < \varepsilon/2.$$

Since $f_2(s) = 0$ for $s \in U$, this shows that $(Pf)f_2 < \varepsilon/2 \mathbb{1}_K$. In particular,

$$\langle \nu_t, (Pf)f_2 \rangle \leq \varepsilon/2 \nu_t(K) \leq \varepsilon/2.$$

On the other hand, since $\|Pf\|_\infty \leq \|f\|_\infty \|P\mathbb{1}_K\|_\infty \leq 1$,

$$\langle \nu_t, (Pf)f_1 \rangle \leq \langle \nu_t, f_1 \rangle \leq \nu_t(U') < \varepsilon/2.$$

Therefore

$$\langle \nu_t, Pf \rangle = \langle \nu_t, (Pf)f_1 \rangle + \langle \nu_t, (Pf)f_2 \rangle < \varepsilon. \quad \square$$

Lemma 4.11. *For every $t \in K$, $\nu_t = 0$.*

Proof. Expanding the right hand side of (4.1), we have the following identity:

$$\sum_{s \in J_t} \lambda(t, s) \delta_s + \nu_t = \sum_{s \in J_t} \sum_{s' \in J_t} \lambda(t, s) \lambda(s, s') \delta_{s'} + \sum_{s \in J_t} \lambda(t, s) \nu_t + P^* \nu_t, \quad (4.2)$$

where we are using that $J_s = J_t$ and also that $\nu_s = \nu_t$ for every $s \in J_t$. Fix $t' \in J_t$. According to Lemma 4.10, for every $\varepsilon > 0$ there exists an open neighborhood V of t' such that $(P^* \nu_t)(V) < \varepsilon$. By making V smaller, we can assume that also $\nu_t(V) < \varepsilon$ and that $t'' \notin V$ for every other $t'' \in J_t$ different from t' . Evaluating the above expression at V yields

$$\lambda(t, t') + \nu_t(V) = \sum_{s \in J_t} \lambda(t, s) \lambda(s, t') + \sum_{s \in J_t} \lambda(t, s) \nu_t(V) + (P^* \nu_t)(V).$$

Using that $\sum_{s \in J_t} \lambda(t, s) \leq 1$, rearrange the previous expression to get

$$\left| \lambda(t, t') - \sum_{s \in J_t} \lambda(t, s) \lambda(s, t') \right| < 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\lambda(t, t') = \sum_{s \in J_t} \lambda(t, s) \lambda(s, t')$.

Using these identities for every $t' \in J_t$, (4.2) becomes

$$\nu_t = \sum_{s \in J_t} \lambda(t, s) \nu_t + P^* \nu_t.$$

Applying P^* to the above equation, and the fact that $\sum_{s \in J_t} \lambda(t, s) > 0$, yields $P^* \nu_t = 0$. Therefore

$$\nu_t = \sum_{s \in J_t} \lambda(t, s) \nu_t$$

and either $\nu_t = 0$, in which case we are done, or $\sum_{s \in J_t} \lambda(t, s) = 1$. But since

$$1 = \mu_t(K) = \sum_{s \in J_t} \lambda(t, s) + \nu_t(K),$$

in the latter case we also end up having $\nu_t = 0$. □

So far we have proved that, for every $t \in K$,

$$\mu_t = \sum_{s \in J_t} \lambda(t, s) \delta_s,$$

where $\alpha \leq \lambda(t, s)$ and J_t is finite with uniformly bounded size $|J_t| \leq 1/\alpha$. Up to this point we have not used that $\alpha \text{id}_{C(K)}$ is the biggest multiplication operator below P , only that $\alpha \text{id}_{C(K)} \leq P$. Now this fact has to come into play.

Lemma 4.12. *For every $n \in \mathbb{N}$, there exists a dense set $D_n \subseteq K$ such that $\lambda(t, t) \leq \alpha + 1/n$ for all $t \in D_n$.*

Proof. Let $n \in \mathbb{N}$. In order to get a contradiction, suppose there exists an open set $U \subseteq K$ such that $\lambda(t, t) > \alpha + 1/n$ for all $t \in U$. Let V be an open set with $\bar{V} \subseteq U$. By Urysohn's lemma, there exists a continuous function $f \in C(K)$ such that $0 \leq f \leq \mathbb{1}_K$, $\text{supp } f \subseteq U$, and $f|_V = 1$.

Set $g = (\alpha + 1/n)f$. Then for every $h \in C(K)_+$ and $t \in U$:

$$g(t)h(t) \leq (\alpha + 1/n)h(t) \leq \lambda(t, t)h(t) \leq (Ph)(t).$$

Also, $g(t)h(t) = 0 \leq (Ph)(t)$ whenever $t \notin U$. Hence $M_g \leq P$, but $\|g\|_\infty = \alpha + 1/n > \alpha$, contradicting the assumption. $\square$

We can now use this fact to obtain the final lemma.

Lemma 4.13. *There exists $t \in K$ such that $\lambda(t, s) = \alpha$ for all $s \in J_t$.*

Proof. Since $|J_s| \leq 1/\alpha$ for all $s \in K$, we can choose $t \in K$ for which the cardinality of J_t is maximal. We claim that $\lambda(t, s) = \alpha$ for all $s \in J_t$. To show this, we will proceed by contradiction: suppose there exist some $t_1 \in J_t$ and some $n \in \mathbb{N}$ for which $\lambda(t, t_1) > \alpha + 1/n$.

Since $J_{t_1} = J_t$, J_{t_1} also has maximal cardinality. Write $J_t = \{t_1, \dots, t_m\}$, and let $U_1, \dots, U_m$ be pairwise disjoint open neighborhoods of $t_1, \dots, t_m$. By the weak* continuity of $s \mapsto \mu_s$, there exists an open neighborhood U of t_1 such that, whenever $s \in U$, $\mu_s(U_1) > \alpha + 1/n$ (recall that $\mu_{t_1}(U_1) > \alpha + 1/n$) and $\mu_s(U_i) > \alpha/2$ (recall that $\mu_{t_1}(U_i) \geq \alpha$) for $i = 2, \dots, m$. This means that, for every $s \in U$, J_s must contain at least a point in each $U_1, \dots, U_m$. Since these sets are pairwise disjoint, J_s contains at least m points. The maximality of m implies then that J_s must contain exactly m points, one in each of the sets $U_1, \dots, U_m$.

In particular, for the $s \in U \cap U_1$, the point of J_s that belongs to U_1 is precisely s . And since $\mu_s(U_1) > \alpha + 1/n$, and the only point of J_s that belongs to U_1 is s , it follows that $\lambda(s, s) > \alpha + 1/n$ for every $s \in U \cap U_1$. This contradicts Lemma 4.12. $\square$

Pick $t \in K$ as in the lemma above. Then

$$\mu_t = \sum_{s \in J_t} \alpha \delta_s$$

so that

$$1 = \mu_t(K) = \sum_{s \in J_t} \alpha = \alpha |J_t|$$

which implies $\alpha = 1/|J_t|$. This completes the proof of Theorem 4.3.

We conclude this section with a general example showing how each of the values 0 and $1/n$, for $n \in \mathbb{N}$, can be attained.

Example 4.14. Let K be a compact Hausdorff space. Let G be a finite subgroup of the homeomorphism group of K . Then the map $P: C(K) \rightarrow C(K)$ defined by

$$Pf = \frac{1}{|G|} \sum_{\sigma \in G} f \circ \sigma, \quad \text{for } f \in C(K),$$

is a positive projection that satisfies the conditions of Theorem 4.3 with $\alpha = 1/|G|$. Indeed, it is immediate to check that P is a positive projection such that $1/|G| \text{id}_{C(K)} \leq P$ and $P1_K = 1_K$. Suppose $0 \leq M_h \leq P$ for some $h \in C(K)_+$. Given $t \in K$, choose an open neighborhood U of t such that $\sigma(t) \notin U$ for all $\sigma \in G \setminus \{\text{id}_K\}$, and let $f \in C(K)$ be a continuous function satisfying $f(t) = 1$, $0 \leq f \leq 1_K$, and $\text{supp } f \subseteq U$. Then

$$h(t) = M_h(f)(t) \leq (Pf)(t) = \frac{1}{|G|}.$$

Since this is true for every $t \in K$, $\|h\|_\infty \leq 1/|G|$.

To construct an example with $\alpha = 0$, let $\mu \in M(K)_+$ be a continuous measure with $\mu(K) = 1$, and define

$$Pf = 1_K \int_K f d\mu \quad \text{for } f \in C(K).$$

It is immediate to check that P is a positive projection satisfying $P1_K = 1_K$. Suppose $0 \leq M_h \leq P$, for some $h \in C(K)_+$. Fix $t \in K$ and $\varepsilon > 0$. Let U be an open neighborhood of t for which $\mu(U) \leq \varepsilon$. Let $f \in C(K)$ be a continuous function satisfying $f(t) = 1$, $0 \leq f \leq 1_K$, and $\text{supp } f \subseteq U$. Then

$$h(t) = M_h(f)(t) \leq (Pf)(t) \leq \mu(U) < \varepsilon.$$

Since $\varepsilon > 0$ and $t \in K$ were arbitrary, $h = 0$.

4.3 WICKSTEAD'S CONJECTURE AND OTHER CONSEQUENCES

Theorem 4.3 can now be used to answer [Wic17c, Question 7.3].

Corollary 4.15. *Let X be a non-zero order complete vector lattice. Let $P: X \rightarrow X$ be a positive projection with constant diagonal α . Then either $\alpha = 0$ or $\alpha = 1/n$, for some $n \in \mathbb{N}$.*

Proof. If $\alpha = 0$ we are done, so assume $\alpha > 0$. In particular, $P \neq 0$, so there exists some $e \in X_+$ such that $Pe \neq 0$. Then I_{Pe} , the ideal generated by Pe , is invariant under P by Lemma 4.4. Use Kakutani's theorem to identify $(I_{Pe}, \|\cdot\|_{Pe})$ lattice isometrically with $C(K)$, for some nonempty compact Hausdorff space K , in such a way that Pe is identified with the function 1_K . We are going to show that the restriction of P to I_{Pe} satisfies the conditions of Theorem 4.3.

It is clear that the restriction is a positive projection satisfying $\alpha \text{id}_{C(K)} \leq P$. Moreover, $P1_K = P(Pe) = Pe = 1_K$. Since X is order complete, and the diagonal of P is αid_X , we can express $P = \alpha \text{id}_X + T$, where T is a positive

operator disjoint from id_X . By the Riesz–Kantorovich formulas, this means that

$$0 = (T \wedge \text{id}_X)(x) = \inf_{0 \leq y \leq x} Ty + (x - y)$$

holds for every $x \in X_+$. Note that $T = P - \alpha \text{id}_X$ also leaves I_{Pe} invariant. Moreover, since I_{Pe} is itself an order complete Banach lattice, the Riesz–Kantorovich formulas also hold in $\mathcal{L}_r(I_{Pe})$. Therefore, for every $x \in (I_{Pe})_+$,

$$(T|_{I_{Pe}} \wedge \text{id}_X)(x) = \inf_{y \in I_{Pe}, 0 \leq y \leq x} Ty + (x - y) = \inf_{0 \leq y \leq x} Ty + (x - y) = 0,$$

where in the second equality we are using that I_{Pe} is an ideal. Hence, in $\mathcal{L}_r(I_{Pe})$, $P|_{I_{Pe}} = \alpha \text{id}_{I_{Pe}} + T|_{I_{Pe}}$, with T disjoint from the identity. Now, if $f \in C(K)_+$ is such that $M_f \leq P$, then because of the previous decomposition and the fact that M_f is a central operator $M_f \leq \alpha \text{id}_{C(K)}$. Hence $\|f\|_\infty \leq \alpha$. $\square$

The previous result applies, in particular, to conditional expectations with constant diagonal. We can use this fact, together with the ideas in the proof of Theorem 4.3, to characterize all contractive positive projections with constant diagonal on ℓ_p spaces. Given a set A , χ_B will denote the characteristic function of $B \subseteq A$, and e_a will denote the characteristic function of the singleton $\{a\} \subseteq A$.

Proposition 4.16. *Let A be a nonempty set, let $1 \leq p < \infty$, and let $P: \ell_p(A) \rightarrow \ell_p(A)$ be a contractive positive projection with constant diagonal $\alpha > 0$. Then $n = 1/\alpha \in \mathbb{N}$, and there exists a partition $\{A_i\}_i$ of A into sets of size n such that*

$$P\left(\sum_{a \in A} \lambda_a e_a\right) = \alpha \sum_i \left(\sum_{a \in A_i} \lambda_a\right) \chi_{A_i},$$

whenever $\sum_{a \in A} |\lambda_a|^p < \infty$. In particular, if A is countable, then there exists an enumeration of A such that the matrix associated with P is

$$\begin{pmatrix} \alpha \mathbf{1}_n & \mathbf{0}_n & \mathbf{0}_n & \cdots \\ \mathbf{0}_n & \alpha \mathbf{1}_n & \mathbf{0}_n & \cdots \\ \mathbf{0}_n & \mathbf{0}_n & \alpha \mathbf{1}_n & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathbf{0}_n$ denotes an n by n block of zeros, and $\mathbf{1}_n$ denotes an n by n block of ones.

Proof. For every $a \in A$, we can express

$$Pe_a = \sum_{b \in J_a} \lambda(a, b) e_b,$$

where $\lambda(a, b) > 0$ and $J_a \subseteq A$ is at most countable. Moreover, that the diagonal of P is α implies $(Pe_a)(a) = \alpha$, that is, $a \in J_a$ and $\lambda(a, a) = \alpha$.

Fix $a \in A$. We claim that, for every $b \in J_a$, $J_b \subseteq J_a$. Indeed, since P is a projection,

$$\sum_{b \in J_a} \lambda(a, b) e_b = P e_a = P(P e_a) = \sum_{b \in J_a} \lambda(a, b) \sum_{c \in J_b} \lambda(b, c) e_c.$$

Since all the coefficients are positive, it cannot be the case that J_b , for $b \in J_a$, introduces a new coordinate. Hence $J_b \subseteq J_a$.

Let I be the closed linear span of $\{e_b : b \in J_a\}$. Notice that I is a closed ideal, and therefore a band. By the above observation and Lemma 4.4, this ideal is invariant under P . We are going to show that the restriction $P|_I : I \rightarrow I$ has rank 1 just like in Lemma 4.8. By Lemma 4.6, it suffices to show that the only invariant closed ideals are the trivial ones. Let $I' \subseteq I$ be a closed ideal invariant under $P|_I$. Since I is order continuous and has a quasi-interior point (in fact, it is lattice isometric to ℓ_p), it follows from Lemma 4.5 that the disjoint complement of I' in I is also invariant. But e_a is an atom: it must belong to either I' or its disjoint complement. Suppose, without loss of generality, that $e_a \in I'$. Then by invariance $P e_a \in I'$ and, being I' an ideal, $e_b \in I'$ for all $b \in J_a$. Since I is the smallest closed ideal containing all the e_b for $b \in J_a$, it follows that $I' = I$. Hence $P|_I$ has rank one.

Let x be a positive norm-one element in $P(I)$. Notice that, in particular, $x(b) > 0$ for all $b \in J_a$. Define a probability measure μ on J_a by $\mu(b) = x(b)^p$, for $b \in J_a$. Then I is lattice isometric to $L_p(\mu)$ through the map

$$\begin{aligned} J: \quad I &\longrightarrow L_p(\mu) \\ y &\longmapsto (y(b)x(b)^{-1})_{b \in J_a}. \end{aligned}$$

Define $\tilde{P} = JP|_I J^{-1} : L_p(\mu) \rightarrow L_p(\mu)$. This is still a contractive positive projection, and it satisfies

$$\tilde{P}(\mathbb{1}_{J_a}) = JPJ^{-1}(\mathbb{1}_{J_a}) = JPx = Jx = \mathbb{1}_{J_a}.$$

By the Andô–Douglas theorem [AA02a, Corollary 5.53], $\tilde{P}$ is the conditional expectation with respect to some σ -algebra Σ on J_a . Since J_a is at most countable, Σ must be generated by a partition $\{A_i\}$ of J_a . Hence the conditional expectation has the form

$$\tilde{P}(f) = \sum_i \chi_{A_i} \frac{1}{\mu(A_i)} \sum_{b \in A_i} \mu(b) f(b) \quad \text{for all } f \in L_p(\mu).$$

From $\tilde{P}(L_p(\mu)) = \text{span}\{\mathbb{1}_{J_a}\}$ it follows that the partition must be trivial, so that actually

$$\tilde{P}(f) = \mathbb{1}_{J_a} \sum_{b \in J_a} \mu(b) f(b) \quad \text{for all } f \in L_p(\mu),$$

and therefore

$$P|_I(y) = J^{-1} \tilde{P} J y = J^{-1}(\mathbb{1}_{J_a}) \sum_{b \in J_a} \mu(b) x(b)^{-1} y(b) = x \sum_{b \in J_a} x(b)^{p-1} y(b)$$

for all $y \in I$. In particular, $\alpha = (Pe_b)(b) = x(b)^p$ for all $b \in J_a$. Since $\alpha > 0$, and $\|x\|_p = 1$, this means that J_a must be finite with $|J_a| = 1/\alpha$. Also, the expression for $P|_I$ ends up being

$$P|_I(y) = \chi_{J_a} \alpha \sum_{b \in J_a} y(b) \quad \text{for all } y \in I. \quad (4.3)$$

Therefore, for every $b \in J_a$, $Pe_b = \alpha \chi_{J_a} = Pe_a$. Hence $J_a = J_b$ whenever $b \in J_a$, and $J_a \cap J_b = \emptyset$ otherwise.

Consider then the partition $\{A_i\}$ formed by the sets of the form J_a , $a \in A$, that are different. Using (4.3) we arrive at

$$P\left(\sum_{a \in A} \lambda_a a\right) = P\left(\sum_i \sum_{a \in A_i} \lambda_a a\right) = \alpha \sum_i \left(\sum_{a \in A_i} \lambda_a\right) \chi_{A_i}$$

whenever $\sum_{a \in A} |\lambda_a|^p < \infty$. In the case that A is at most countable, to get the desired matrix first enumerate the partition $\{A_k\}_{k \in \mathbb{N}}$, and then enumerate A in such a way that $A_k = \{n(k-1) + 1, \dots, nk\}$, where $n = 1/\alpha$. $\square$

Remark 4.17. Following the ideas of the previous proof, it is not difficult to check that the only contractive positive projection $P: \ell_p(A) \rightarrow \ell_p(A)$ with constant diagonal 0 is $P = 0$.

4.4 ON THE REPRESENTATION OF BANACH LATTICE ALGEBRAS

In this section we use Corollary 4.15 to settle the representation problem for Banach lattice algebras in the negative. We begin with two preparatory lemmas: the first records that the range of a positive projection is itself an order complete vector lattice, and the second shows that disjointness from a positive idempotent in $\mathcal{L}_r(X)$ becomes disjointness from the identity on its range. These lemmas will let us apply Corollary 4.15 on the range of an arbitrary positive idempotent of $\mathcal{L}_r(X)$.

In the following lemmas, when X is a vector lattice, Y is a sublattice, and $S \subseteq Y$ is a set, we denote by $\sup_X S$ the supremum of S in X (whenever it exists) and by $\sup_Y S$ the supremum of S in Y (whenever it exists).

Lemma 4.18. *Let X be an order complete vector lattice, and let $E: X \rightarrow X$ be a positive projection. Then $Y = E(X)$, endowed with the order inherited from X , is an order complete vector lattice with supremum $\vee_Y$ given by*

$$y_1 \vee_Y y_2 = E(y_1 \vee y_2) \quad \text{for all } y_1, y_2 \in Y.$$

Proof. Let $y_1, y_2 \in Y$. Since E is positive and $Ey_i = y_i$ for $i = 1, 2$, we have $E(y_1 \vee y_2) \geq Ey_i = y_i$, so $E(y_1 \vee y_2)$ is an upper bound of $\{y_1, y_2\}$ in Y . If $z \in Y$ satisfies $z \geq y_1$ and $z \geq y_2$, then $z \geq y_1 \vee y_2$ in X , whence $z = Ez \geq E(y_1 \vee y_2)$. This shows that $y_1 \vee_Y y_2 = E(y_1 \vee y_2)$.

For order completeness, let (y_γ) be an increasing net in Y that is bounded above by some $y_0 \in Y$. By order completeness of X , $z = \sup_X y_\gamma$ exists. Then

$Ez \geq Ey_\gamma = y_\gamma$ for all γ , so Ez is an upper bound of (y_γ) in Y . If $w \in Y$ also satisfies $w \geq y_\gamma$ for all γ , then $w \geq z$ in X , and so $w = Ew \geq Ez$. Hence $Ez = \sup_Y y_\gamma$. $\square$

Lemma 4.19. *Let X be an order complete vector lattice, and let $E, T \in \mathcal{L}_r(X)$ be positive operators satisfying:*

- (i) $E^2 = E$,
- (ii) $ET = TE = T$, and
- (iii) $E \wedge T = 0$ in $\mathcal{L}_r(X)$.

Set $Y = E(X)$, endowed with the order inherited from X . Then

$$\text{id}_Y \wedge T|_Y = 0 \quad \text{in } \mathcal{L}_r(Y).$$

Proof. By Lemma 4.18, Y is an order complete vector lattice, so $\mathcal{L}_r(Y)$ is itself an order complete vector lattice and the Riesz–Kantorovich formulas hold in it. Note also that $T = ET = TE$ implies that T leaves Y invariant, so the restriction $T|_Y : Y \rightarrow Y$ is a well-defined element of $\mathcal{L}_r(Y)$.

Fix $y \in Y_+$. We compute $(E \wedge T)(y)$ using the Riesz–Kantorovich formula in $\mathcal{L}_r(X)$:

$$(E \wedge T)(y) = \inf_X \{ Ea + T(y - a) : 0 \leq a \leq y \}.$$

Since $T = TE$, for any $a \in X_+$ with $a \leq y$,

$$Ea + T(y - a) = Ea + Ty - TEa = (\text{id}_Y - T|_Y)(Ea) + T|_Y(y).$$

As a ranges over $\{a \in X_+ : a \leq y\}$, the element $w = Ea$ ranges over the order interval $[0, y]_Y = \{w \in Y_+ : w \leq y\}$ in Y . Therefore

$$(E \wedge T)(y) = \inf_X \{ (\text{id}_Y - T|_Y)(w) + T|_Y(y) : w \in [0, y]_Y \}. \quad (4.4)$$

On the other hand, the Riesz–Kantorovich formula in $\mathcal{L}_r(Y)$ gives

$$(\text{id}_Y \wedge T|_Y)(y) = \inf_Y \{ (\text{id}_Y - T|_Y)(w) + T|_Y(y) : w \in [0, y]_Y \}. \quad (4.5)$$

The set whose infimum is taken is the same subset of Y in both (4.4) and (4.5). The only difference is that the first infimum is taken in X while the second is taken in $Y \subseteq X$. Therefore

$$(\text{id}_Y \wedge T|_Y)(y) \leq (E \wedge T)(y) = 0.$$

Since this holds for every $y \in Y_+$, we conclude that $\text{id}_Y \wedge T|_Y = 0$ in $\mathcal{L}_r(Y)$. $\square$

We can now state the criterion for non-representability. The proposition below generalizes the criterion implicit in [Wic17c, Section 7] in two ways: the algebra A need not have an algebraic identity, and the conclusion does not require the representation to be unital.

Proposition 4.20. *Let A be a lattice-ordered algebra. Suppose there exist non-zero positive elements $e, p \in A_+$ satisfying:*

$$(i) \quad e^2 = e, \quad p^2 = p,$$

$$(ii) \quad ep = pe = p, \text{ and}$$

$$(iii) \quad p = \alpha e + x, \text{ for some } x \in A \text{ with } x \wedge e = 0, \text{ and some } \alpha \in \mathbb{R}_+.$$

If $\alpha \notin \{0\} \cup \{1/n : n \in \mathbb{N}\}$, then A has no faithful representation on $\mathcal{L}_r(X)$, for X any order complete vector lattice.

Proof. Suppose that $\Phi: A \rightarrow \mathcal{L}_r(X)$ is a faithful representation of A , where X is an order complete vector lattice. Set $E = \Phi(e)$, $P = \Phi(p)$, and $T = \Phi(x)$. Since Φ is a lattice and algebra homomorphism, E and P are positive projections, and they are non-zero because Φ is faithful and $e, p \neq 0$. The relations

$$ex = xe = x,$$

which follow from $ep = pe = p$ together with $p = \alpha e + x$, give

$$ET = TE = T.$$

Applying Φ to (iii) yields $P = \alpha E + T$, and the disjointness $e \wedge x = 0$ in A transfers via Φ to $E \wedge T = 0$ in $\mathcal{L}_r(X)$.

By Lemma 4.19, setting $Y = E(X)$, we obtain $\text{id}_Y \wedge T|_Y = 0$ in $\mathcal{L}_r(Y)$. Restricting the equality $P = \alpha E + T$ to Y yields

$$P|_Y = \alpha \text{id}_Y + T|_Y,$$

where, by Lemma 4.18, Y is an order complete vector lattice (non-zero, since $E \neq 0$), $P|_Y$ is a positive projection on it, and $T|_Y$ is disjoint from id_Y . That is, $P|_Y$ has constant diagonal α . By Corollary 4.15, $\alpha \in \{0\} \cup \{1/n : n \in \mathbb{N}\}$. $\square$

Remark 4.21. When e is an algebraic identity and Φ is unital, Lemma 4.19 is not actually needed: Φ being unital then forces $E = \text{id}_X$, so $Y = X$, and the operator $P = \Phi(p)$ already has constant diagonal α in $\mathcal{L}_r(X)$, allowing one to apply Corollary 4.15 directly. The role of Lemma 4.19 is precisely to handle the case in which $E = \Phi(e)$ is not the identity of $\mathcal{L}_r(X)$, which is what one encounters either when the algebra has no identity or when the representation is not required to be unital.

The following example, due to Wickstead [Wic17c, Section 7], exhibits Banach lattice algebras of dimension 2 satisfying the conditions of Proposition 4.20. In its original form it answered [Wic17a, Question 5.3] and [Wic17b, Problem 2] in the case of unital representations; with the strengthened proposition above, the conclusion no longer depends on the representation being unital.

Example 4.22. Let $-1 \leq \beta \leq 0$. Consider $\mathbb{R}^2$ with pointwise multiplication and positive cone

$$P_\beta^1 = \{ (x, y) \in \mathbb{R}^2 : 0 \leq x, \beta x \leq y \leq x \}.$$

It is shown in [Wic17c, Theorem 3.1] that this is a lattice cone for which the product of positive elements is positive. Moreover, $(\mathbb{R}^2, P_\beta^1)$ can be endowed with a norm so as to make it a Banach lattice algebra [Wic17c, Proposition 8.3]. Also, the algebraic identity $e = (1, 1)$ is positive.

Let $p = (1, 0)$ and $x = (1, \beta)$. Note that $p^2 = p$, $p \geq 0$, and that

$$p = \frac{\beta}{\beta - 1}e + \frac{1}{1 - \beta}x.$$

Let us check that $x \wedge e = 0$. If $y = (y_1, y_2)$ and $x, e \geq y$, then

$$\beta(1 - y_1) \leq \beta - y_2 \leq 1 - y_1 \text{ and } \beta(1 - y_1) \leq 1 - y_2 \leq 1 - y_1,$$

which give $\beta y_1 \geq y_2 \geq y_1$. Thus $y_1 \leq 0$, which together with $\beta y_1 \geq y_2 \geq y_1$, implies that $y \leq 0$.

By Proposition 4.20, applied with $e = (1, 1)$ and $p = (1, 0)$, if

$$\frac{\beta}{\beta - 1} \notin \{0\} \cup \{1/n : n \in \mathbb{N}\},$$

that is, if $\beta \notin \{0\} \cup \{-1/(n - 1) : n \in \mathbb{N}, n \geq 2\}$, then $(\mathbb{R}^2, P_\beta^1)$ admits no faithful representation on any order complete vector lattice.

5 REPRESENTATION THEORY

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The previous chapter settled the representation problem for Banach lattice algebras in the negative: we exhibited Banach lattice algebras with identity of dimension 2 that admit no faithful lattice and algebra homomorphism into $\mathcal{L}_r(X)$, for any order complete vector lattice X . This rules out the hope of a universal, GNS-style representation theorem embedding every Banach lattice algebra into a concrete algebra of regular operators. It does not, however, rule out a representation theory. What closes with that result is one particular route to the study of lattice-ordered algebras—via a concrete ambient $\mathcal{L}_r(X)$ —but the intrinsic representation-theoretic questions, those concerning modules, submodules, quotients, and the structure of semisimple objects, are untouched by it. The present chapter takes up those questions systematically. All the contents in this chapter are joint work with M. de Jeu, and will be part of a manuscript that is still in preparation.

For associative algebras, the analogous theory is classical and well developed: modules and submodules, quotients, simple and semisimple modules, Schur’s lemma, the Wedderburn–Artin classification of semisimple algebras with identity, Fitting’s lemma and the Krull–Schmidt theorem on indecomposable modules fit together into a coherent body of results (see, for instance, [EH18]). For lattice-ordered algebras, by contrast, very little is known. Some preliminary work on representations of groups on vector lattices is available and supplies a shared vocabulary, but the corresponding programme for algebras has not been carried out.

Already at the level of the basic definitions it becomes apparent that the interaction between the lattice and algebra structures forces each classical notion to split into several distinct grades. A representation of a lattice-ordered algebra A on a vector lattice X is, in the weakest sense, an algebra homomorphism $\phi: A \rightarrow \mathcal{L}(X)$ that happens to be regular; one obtains progressively stronger notions by requiring ϕ to be positive, or, when X is order complete, a lattice homomorphism into $\mathcal{L}_r(X)$. Submodules, in turn, admit several natural choices: A -invariant subspaces, sublattices, ideals, and bands. Quotients are well behaved modulo A -invariant ideals in the positive case, and modulo A -invariant bands in the regular and lattice cases. These distinctions are not cosmetic: whether an algebraic result survives the passage to the ordered setting depends on which grade is taken.

The chapter is organised as follows. Section 5.1 introduces the three grades of ordered module—regular, positive, and lattice—and collects the basic examples. Among these we include a bounded algebra homomorphism from a Banach lattice algebra into a space of regular operators that is not itself regular, answering a question raised in [Bla25, Page 7]. Section 5.2 studies A -invariant ideals and bands, and the associated quotients, direct sums and products. Section 5.3 introduces the ordered analogues of simplicity and semisimplicity, formulated in terms of A -invariant ideals, and characterizes semisimple modules by the existence of invariant complements (Theorem 5.11). Section 5.4 gives an ordered version of Schur’s lemma, and Section 5.5 an ordered version of the Wedderburn–Artin theorem (Theorem 5.27). In the normed case the latter takes a particularly clean form: every order semisimple normed lattice-ordered algebra with identity is lattice-algebra isomorphic to a finite product of matrix algebras over $\mathbb{R}$, and in particular admits a faithful lattice-algebra representation on some finite dimensional vector lattice (Corollary 5.30). For the algebras in this class, the representation problem has a positive answer after all.

5.1 ORDERED MODULES

Let A be a real algebra. Recall that an A -module is a real vector space V together with an algebra homomorphism $\phi: A \rightarrow \mathcal{L}(V)$, where $\mathcal{L}(V)$ denotes the linear operators on V . Such a map is also called an *algebraic representation of A on V* . When no confusion can arise we omit the explicit mention of ϕ and denote $\phi(a)(x)$ simply by ax . When A has an identity 1_A , we adopt the convention that $\phi(1_A)$ must be the identity map on V .

Our goal is to study A -modules in the ordered context. A direct way to do so is to work with a vector lattice X that also has an A -module structure. For some results this will suffice. For others, we need the order structure to also appear in A (making it into a lattice-ordered algebra) and in ϕ . There are several natural ways to do so:

Definition 5.1. Let A be a lattice-ordered algebra, let X be a vector lattice and let $\phi: A \rightarrow \mathcal{L}(X)$ be an algebra homomorphism.

- (i) We say that X is a *regular A -module* if ϕ is regular. In this case, ϕ is called a *regular representation of A on X* .
- (ii) We say that X is a *positive A -module* if ϕ is positive. In this case, ϕ is called a *positive representation of A on X* .
- (iii) Suppose that X is order complete. We say that X is a *lattice A -module* if $\phi(A) \subseteq \mathcal{L}_r(X)$ and $\phi: A \rightarrow \mathcal{L}_r(X)$ is a lattice homomorphism. In this case, ϕ is called a *lattice representation of A on X* .

Note that for regular and positive representations, $\phi(a)$ is automatically a regular operator. In the case that A is a Banach lattice algebra and X is a Banach lattice, we speak of *regular/positive/lattice Banach A -modules*. In these

cases, X is a Banach A -module, in the sense that $\phi(a)$ is a bounded operator on X for every $a \in A$ and the map ϕ is itself bounded. According to [Bla25, Section 2], the notion of regular Banach A -module was first introduced by M. Roelands (with the name regular Banach lattice A -module) and the notion of positive Banach A -module by the author himself (with the name Banach lattice A -module). Here we adopt a different nomenclature to make it more consistent.

Every lattice A -module is a positive A -module and every positive A -module is a regular A -module. It is easy to see that each of these notions is strictly stronger than the next. However, is every A -module satisfying $\phi(A) \subseteq \mathcal{L}_r(X)$ a regular A -module? In other words: is every algebra homomorphism $\phi: A \rightarrow \mathcal{L}_r(X)$ regular? At least in the bounded case, this was an open question. In [Bla25, Page 7] it is mentioned that no example is known of a continuous algebra homomorphism from a Banach lattice algebra A to a Banach lattice algebra of the form $\mathcal{L}_r(X)$, with X an order complete Banach lattice, which is not regular. The following is an example of such a map.

Example 5.2. This example is partially based on Krengel's example of a compact operator that does not have a modulus. Let us recall the construction following [AB06, Example 5.6]. Let $X_n = \ell_2^{2^n}$ for $n = 0, 1, 2, \dots$. Identify X_n with $X_{n-1} \oplus_2 X_{n-1}$ in the natural way. Define $T_n: X_n \rightarrow X_n$ inductively as follows: T_0 is the identity operator and

$$T_n = \frac{1}{\sqrt{2}} \begin{pmatrix} T_{n-1} & T_{n-1} \\ -T_{n-1} & T_{n-1} \end{pmatrix}$$

for $n \geq 1$. The first three operators of the sequence have matrices:

$$T_0 = (1), \quad T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad T_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It can be shown by induction on n that each T_n is an isometry.

Identify X_n with the subspace $\text{span}\{e_{2^n}, \dots, e_{2^{n+1}-1}\}$ of ℓ_2 , that is, embed X_n in ℓ_2 in the coordinates that go from 2^n to $2^{n+1}-1$. All of these identifications together yield a decomposition of ℓ_2 into blocks $\oplus_{n=0}^{\infty} X_n$, where the direct sum has the ℓ_2 -norm. Define the block-diagonal operator $T = \oplus_{n=0}^{\infty} T_n$. Again, it is not difficult to check that T is an isometry.

If T had a modulus, the matrix entries of the modulus would have to be at least $|\langle e_i, T e_j \rangle|$, where $\langle \cdot, \cdot \rangle$ denotes the inner product of ℓ_2 . However, the matrix $(|\langle e_i, T e_j \rangle|)_{i,j}$ does not define an operator on ℓ_2 . Indeed, define $u_n = e_{2^n} + \dots + e_{2^{n+1}-1}$. Then $u = \oplus_{n=0}^{\infty} 2^{-2n/3} u_n$ is an element of ℓ_2 whose image through $(|\langle e_i, T e_j \rangle|)_{i,j}$ would be

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |\langle e_i, T e_j \rangle| |\langle u, e_j \rangle| \right) e_i = \bigoplus_{n=0}^{\infty} 2^{n/2} 2^{-2n/3} u_n = \bigoplus_{n=0}^{\infty} 2^{-n/6} u_n. \quad (5.1)$$

Since $\|u_n\|_2^2 = 2^n$, this sequence does not define an element of ℓ_2 .

We are now going to construct an isometric embedding $S: \ell_2 \rightarrow \mathcal{L}_r(\ell_2 \oplus \ell_2)$ such that the product of any two elements in $S(\ell_2)$ is zero. This idea works more generally to give an embedding from a Banach lattice X to $\mathcal{L}_r(X^* \oplus X^*)$ with the stated property.

For every $x \in \ell_2$, define

$$S_x: \begin{array}{ccc} \ell_2 \oplus \ell_2 & \longrightarrow & \ell_2 \oplus \ell_2 \\ (a, b) & \longmapsto & (0, \langle a, x \rangle e_1) \end{array}.$$

Note that this is a regular operator. The map $S: \ell_2 \rightarrow \mathcal{L}_r(\ell_2 \oplus \ell_2)$ that sends $x \in \ell_2$ to S_x is a positive linear map (it is, in fact, an isometric embedding). Moreover, if ℓ_2 is endowed with the zero product (which makes it into a Banach lattice algebra), then S is an algebra homomorphism, because $S_x S_y = 0$ for all $x, y \in \ell_2$.

Consider the operator $ST: \ell_2 \rightarrow \mathcal{L}_r(\ell_2 \oplus \ell_2)$. This is an algebra homomorphism as long as ℓ_2 is equipped with the zero product. It is also an isometric embedding. We are going to show that ST is not regular.

If it were, there would exist a positive operator $A: \ell_2 \rightarrow \mathcal{L}_r(\ell_2 \oplus \ell_2)$ such that $\pm ST \leq A$. This would mean that for all $x, a \in (\ell_2)_+$:

$$(Ax)(a, 0) \geq \pm(STx)(a, 0) = (0, \pm \langle a, Tx \rangle e_1).$$

Let $P: \ell_2 \oplus \ell_2 \rightarrow \ell_2$ be the projection onto the second coordinate. Then

$$\langle P[(Ax)(a, 0)], e_1 \rangle \geq |\langle a, Tx \rangle|. \quad (5.2)$$

Observe that the map

$$B: \begin{array}{ccc} \ell_2 \times \ell_2 & \longrightarrow & \mathbb{R} \\ (x, a) & \longmapsto & \langle P[(Ax)(a, 0)], e_1 \rangle \end{array}$$

is bilinear and positive (in the sense that $x, a \geq 0$ implies $B(x, a) \geq 0$). Inequality (5.2) yields $B(e_i, e_j) \geq |\langle e_i, Te_j \rangle|$. Let $N \in \mathbb{N}$, and put $M = 2^N - 1$. Using that $u \geq 0$, compute:

$$\begin{aligned} B(u, u) &\geq B\left(\sum_{i=1}^M \langle u, e_i \rangle e_i, \sum_{j=1}^M \langle u, e_j \rangle e_j\right) \\ &= \sum_{i=1}^M \sum_{j=1}^M B(e_i, e_j) \langle u, e_i \rangle \langle u, e_j \rangle \\ &\geq \sum_{i=1}^M \sum_{j=1}^M |\langle e_i, Te_j \rangle| \langle u, e_i \rangle \langle u, e_j \rangle \\ &= \left\langle u, \sum_{i=1}^M \sum_{j=1}^M |\langle e_i, Te_j \rangle| \langle u, e_j \rangle e_i \right\rangle \\ &= \left\langle \bigoplus_{n=0}^{\infty} 2^{-2n/3} u_n, \bigoplus_{n=0}^N 2^{-n/6} u_n \right\rangle, \end{aligned}$$

where in the last equality we have used the definition of u and (5.1) truncated to the first $M = 2^{N+1} - 1$ coordinates (i.e., to the first N blocks). Computing the last inner product yields:

$$B(u, u) \geq \sum_{n=0}^N 2^{-n/6} 2^{-2n/3} 2^n = \sum_{n=0}^N 2^{n/6}.$$

Since $N \in \mathbb{N}$ was arbitrary, this is a contradiction.

Now for some examples of ordered modules. This first example collects the “natural” module structures.

- Example 5.3.** (i) Every lattice-ordered algebra A is a positive A -module through the left regular representation $L: A \rightarrow \mathcal{L}_r(A)$, where $L(a)(b) = ab$ for all $a, b \in A$.
- (ii) Let X be a vector lattice and let A be a subalgebra of $\mathcal{L}_r(X)$. Suppose that A is also a lattice with the operator order. Then X has a natural positive A -module structure given by the inclusion map of A in $\mathcal{L}_r(X)$. When X is order complete, and A is also a sublattice, this map makes X into a lattice A -module.
- (iii) Scalar multiplication makes every vector lattice into a lattice $\mathbb{R}$ -module. More precisely, this module structure is provided by the representation $\phi: \mathbb{R} \rightarrow \mathcal{L}_r(X)$, where $\phi(\lambda) = \lambda \text{id}_X$.

The following is also a standard example that will be useful later.

Example 5.4. Consider the algebra of polynomials $\mathbb{R}[x]$ with positive cone

$$\mathbb{R}[x]_+ = \mathbb{R}_+[x] = \left\{ \sum_{i=0}^n a_i x^i : n \in \mathbb{N}, a_0, \dots, a_n \in \mathbb{R}_+ \right\}.$$

This is a lattice-ordered algebra. Let X be a vector lattice and let $T: X \rightarrow X$ be a regular operator. Then there exists a unique algebra homomorphism $\phi: \mathbb{R}[x] \rightarrow \mathcal{L}_r(X)$ satisfying $\phi(x) = T$ and $\phi(1) = \text{id}_X$. This map makes X into a regular $\mathbb{R}[x]$ -module. Indeed, if $T = T_+ - T_-$, with $T_+, T_- \in \mathcal{L}_r(X)$, then $\phi(x^n) = (T_+ - T_-)^n$ is a regular operator for all $n \in \mathbb{N}$. Let $\phi(x^n)_+$ and $\phi(x^n)_-$ be positive and such that $\phi(x^n) = \phi(x^n)_+ - \phi(x^n)_-$. Then the unique positive linear maps $\phi_+, \phi_-: \mathbb{R}[x] \rightarrow \mathcal{L}_r(X)$ defined by $\phi_+(x^n) = \phi(x^n)_+$, $\phi_-(x^n) = \phi(x^n)_-$, and $\phi_+(1) = \text{id}_X$, $\phi_-(1) = 0$ are such that $\phi = \phi_+ - \phi_-$. Note that X is a positive $\mathbb{R}[x]$ -module if and only if T is positive.

The final example is characteristic of Banach lattice algebras and appears naturally in the theory.

Example 5.5. Let A be a Banach lattice algebra with an identity $e \in A$ of norm one. As seen in Chapter 3, the ideal A_e generated by e is a projection band and a subalgebra that, with the norm induced by e , is lattice-algebra isometric to

$C(K)$, for some compact Hausdorff space K . Its disjoint complement A_e^d need not be a subalgebra. However, if $x \in A_e$ and $y \in A_e^d$, then $xy \in A_e^d$ because

$$|xy| \wedge e \leq (|x||y|) \wedge e \leq (\|x\|_e |y|) \wedge e = (\|x\|_e |y|) \wedge e = 0.$$

Hence we can restrict the left multiplication operators $L_x: A_e^d \rightarrow A_e^d$. Then the map $L: A_e \rightarrow \mathcal{L}_r(A_e^d)$, $Lx = L_x$, makes A_e^d into a positive A_e -module.

5.2 INVARIANT SUBSTRUCTURES, QUOTIENTS AND PRODUCTS

Let A be a lattice-ordered algebra, let X be a vector lattice, and let $\phi: A \rightarrow \mathcal{L}(X)$ be an algebraic representation. A subset S of X is said to be *A-invariant* if $\phi(a)(S) \subseteq S$ for all $a \in A$. If we denote, as usual, the action of a family of operators $\mathcal{F} \subseteq \mathcal{L}(X)$ on a set $S \subseteq X$ by

$$\mathcal{F}(S) = \{Tx : T \in \mathcal{F}, x \in S\},$$

then S is A -invariant if and only if $\phi(A)(S) \subseteq S$. An A -invariant vector subspace is also called a *submodule*. For every A -invariant subspace Y there is an associated algebraic representation $A \rightarrow \mathcal{L}(Y)$ sending $a \in A$ to $\phi(a)|_Y$.

Since X is a vector lattice, there are three other common substructures of X : sublattices, ideals, and bands. This section is devoted to the A -invariant versions of ideals and bands, and the quotients associated with them. We start with some examples.

Example 5.6. (i) Let X be a vector lattice. With the natural $\mathbb{R}$ -module structure on X all subspaces, sublattices, ideals and bands are $\mathbb{R}$ -invariant.

(ii) Let K be a compact Hausdorff space. If X is a positive $C(K)$ -module, then every ideal is $C(K)$ -invariant. Indeed, denote by $\phi: C(K) \rightarrow \mathcal{L}_r(X)$ the positive representation of $C(K)$ on X . Then ϕ maps the constant one function to the identity map of X . Positivity of ϕ implies that $\phi(C(K))$ is a subspace of the center of X . Since central operators leave ideals invariant, every ideal is $C(K)$ -invariant.

More generally, we have the following.

Proposition 5.7. *Let A be a lattice-ordered algebra, let X be a vector lattice and let $\phi: A \rightarrow \mathcal{L}_r(X)$ be an algebraic representation.*

- (i) *If $\phi(A) \subseteq \mathcal{Z}(X)$, then every ideal is A -invariant.*
- (ii) *Suppose X is order complete. If $\mathcal{Z}(X) \subseteq \phi(A)$, then every A -invariant sublattice is an ideal.*
- (iii) *Suppose X is order complete. If $\phi(A) = \mathcal{Z}(X)$, then ideals and A -invariant sublattices coincide.*

Proof. (i) It follows from the fact that ideals are invariant under central operators.

- (ii) According to [AB06, Theorem 2.49], for every $x \in X_+$ we have $\mathcal{Z}(X)x = I_x$, where I_x denotes the principal order ideal generated by x . Let Y be an A -invariant sublattice. For every $x \in Y_+$:

$$I_x = \mathcal{Z}(X)x \subseteq \phi(A)x \subseteq Y.$$

This shows that Y is an ideal.

- (iii) It follows from (i) and (ii). $\square$

For every A -invariant sublattice, ideal or band Y there is an associated algebraic representation $A \rightarrow \mathcal{L}(Y)$ sending $a \in A$ to $\phi(a)|_Y$. This algebraic representation is regular (resp. positive) as long as ϕ is. The case of lattice representations is treated in the upcoming proposition.

If A is a real algebra, and $\phi: A \rightarrow \mathcal{L}(X)$ is an algebraic representation of A on a vector space X , then for every A -invariant subspace $Y \subseteq X$ the map $\tilde{\phi}: A \rightarrow \mathcal{L}(X/Y)$ defined by

$$\tilde{\phi}(a)(x + Y) = \phi(a)(x) + Y \quad \text{for all } a \in A, x \in X, \quad (5.3)$$

is an algebraic representation of A on X/Y . The space X/Y , with this particular module structure, is called the *quotient module of X modulo Y* . In the case of ordered modules, the following can be said about quotients.

Proposition 5.8. *Let A be a lattice-ordered algebra, let X be a vector lattice, and let $\phi: A \rightarrow \mathcal{L}(X)$ be an algebraic representation of A on X . Let $Y \subseteq X$ be an A -invariant subspace, and let $\tilde{\phi}: A \rightarrow \mathcal{L}(X/Y)$ be the map defined in (5.3).*

- (i) *Suppose ϕ is a positive representation. If Y is an A -invariant ideal, then $\tilde{\phi}: A \rightarrow \mathcal{L}_r(X/Y)$ is a positive representation of A on X/Y .*
- (ii) *Suppose X is order complete and ϕ is a regular representation. If Y is an A -invariant band, then $\tilde{\phi}: A \rightarrow \mathcal{L}_r(X/Y)$ is a regular representation of A on X/Y .*
- (iii) *Suppose X is order complete and ϕ is a lattice representation. If Y is an A -invariant band, then $\tilde{\phi}: A \rightarrow \mathcal{L}_r(X/Y)$ is a lattice representation of A on X/Y . Moreover, the representation $A \rightarrow \mathcal{L}_r(Y)$ given by restriction of operators is also a lattice representation.*

Proof. (i) We need only check that $\tilde{\phi}$ is positive. Let $a \in A_+$ and let $x + Y \in (X/Y)_+$. There exists $y \in X_+$ such that $y + Y = x + Y$. Since ϕ is positive, $\phi(a)(y) \in X_+$, and therefore

$$\tilde{\phi}(a)(x + Y) = \tilde{\phi}(a)(y + Y) = \phi(a)(y) + Y \in (X/Y)_+.$$

- (ii) Since we are assuming X to be order complete, so is $\mathcal{L}_r(X)$, and therefore $|\phi|$ exists and is given by the Riesz–Kantorovich formula. We want to show that Y is invariant under $|\phi|(a)$ for every $a \in A_+$. For this, in turn,

we need to show that Y is invariant under $|\phi(b)|$ for every $b \in A$. If $x \in Y_+$, then

$$|\phi(b)|(x) = \sup_{|y| \leq x} |\phi(b)(y)|.$$

Since $x \in Y_+$, we also have $y \in Y_+$ for every $|y| \leq x$; A -invariance of Y implies that $\phi(b)(y) \in Y$. Using that Y is a band it follows that $|\phi(b)|(x) \in Y_+$.

Fix $a \in A_+$. According to [AB06, Theorem 1.21], we can express $|\phi|(a)$ as the supremum of an increasing net:

$$|\phi|(a) = \sup \left\{ \sum_{i=1}^n |\phi(b_i)| : n \in \mathbb{N}, b_1, \dots, b_n \in A_+, \sum_{i=1}^n b_i = a \right\}.$$

Recall from [AB06, Theorem 1.19] that the supremum of an increasing net of operators on an order complete vector lattice can be computed pointwise on positive elements. Hence, for every $x \in Y_+$,

$$|\phi|(a)(x) = \sup \left\{ \sum_{i=1}^n |\phi(b_i)|(x) : n \in \mathbb{N}, b_1, \dots, b_n \in A_+, \sum_{i=1}^n b_i = a \right\}.$$

We have already observed that $|\phi(b_i)|(x) \in Y_+$. Since Y is a band, the supremum defining $|\phi|(a)(x)$ also belongs to Y_+ .

This completes the proof that Y is invariant under $|\phi|(a)$ for every $a \in A_+$ (and, in fact, for every $a \in A$). Since Y is also invariant under $\phi(a)$, adding and subtracting these quantities it follows that it must also be invariant under $\phi_+(a)$ and $\phi_-(a)$ for every $a \in A$. Even though ϕ_+ and ϕ_- need not be algebraic representations, it is immediate to check using the argument of (i) that the maps $\tilde{\phi}_+, \tilde{\phi}_- : A \rightarrow \mathcal{L}_r(X/Y)$, defined as in (5.3), are well-defined positive maps. It is also direct that $\tilde{\phi} = \tilde{\phi}_+ - \tilde{\phi}_-$. Therefore $\tilde{\phi}(A) \subseteq \mathcal{L}_r(X/Y)$ and $\tilde{\phi}$ is a regular representation of A on X/Y .

- (iii) First note that the image of $\tilde{\phi}$ is indeed in $\mathcal{L}_r(X/Y)$ because $\tilde{\phi}$ is positive by (i). Being X order complete, Y is a projection band, and the quotient X/Y is again order complete (it is, in fact, lattice isomorphic to Y^d). Moreover, the quotient map $q : X \rightarrow X/Y$ is an order continuous lattice homomorphism; in particular, it preserves arbitrary suprema. With these facts, we can now use the Riesz–Kantorovich formulas to show that $\tilde{\phi}$ is

a lattice homomorphism: if $a \in A$ and $x \in X_+$, then

$$\begin{aligned}
 |\tilde{\phi}(a)|(q(x)) &= \sup_{|q(y)| \leq q(x)} |\tilde{\phi}(a)(q(y))| \\
 &= \sup_{|q(y)| \leq q(x)} |q(\phi(a)(y))| \\
 &= \sup_{|q(y)| \leq q(x)} q(|\phi(a)(y)|) \\
 &= q\left(\sup_{|q(y)| \leq q(x)} |\phi(a)(y)|\right) \\
 &\geq q\left(\sup_{|y| \leq x} |\phi(a)(y)|\right) \\
 &= q(|\phi(a)|(x)) \\
 &= q(\phi(|a|)(x)) \\
 &= \tilde{\phi}(|a|)(q(x)).
 \end{aligned}$$

Since this holds for every element of $(X/Y)_+$, it follows that $|\tilde{\phi}(a)| \geq \tilde{\phi}(|a|)$. The reverse inequality is a consequence of the positivity of $\tilde{\phi}$. Hence $\tilde{\phi}$ is a lattice homomorphism.

Finally, it is direct from the Riesz–Kantorovich formulas that the modulus of $\phi(a)|_Y$ in $\mathcal{L}_r(Y)$ is precisely $(|\phi(a)|)|_Y = \phi(|a|)|_Y$. $\square$

If X is an order complete vector lattice and $Y \subseteq X$ is a band, then X/Y is lattice isomorphic to Y^d . We have shown in the previous proposition that, whenever X has a lattice A -module structure, so does X/Y . However, this does not mean that Y^d is A -invariant, as the following example shows.

Example 5.9. Let A be the sublattice-algebra of $\mathbb{R}[x]$ consisting of the polynomials with zero constant term. Consider the unique algebra homomorphism $\phi: A \rightarrow \mathcal{L}(\mathbb{R}^2)$ defined by $\phi(x)(\lambda, \mu) = (\mu, 0)$ for all $(\lambda, \mu) \in \mathbb{R}^2$. Let us check that this is a lattice homomorphism. Note first that $\phi(x^2) = \phi(x)^2 = 0$. Let $p = a_1x + \cdots + a_nx^n \in A$. We then have $\phi(p) = a_1\phi(x)$, and since $\phi(x) \geq 0$, it follows that

$$|\phi(p)| = |a_1|\phi(x) = \phi(|a_1|x).$$

Similarly,

$$\phi(|p|) = \phi(|a_1|x + \cdots + |a_n|x^n) = \phi(|a_1|x).$$

Hence $\mathbb{R}^2$ is a lattice A -module. Denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$. Note that $\phi(x)(e_1) = 0$, so that the span of e_1 is an A -invariant band. Its disjoint complement is the span of e_2 , which is not A -invariant because $\phi(x)(e_2) = e_1$.

The A -modules in which every A -invariant ideal is a projection band and has an A -invariant complement are central to the theory and will be studied in detail in Section 5.3. In preparation, we first have to introduce products and direct sums.

Let A be an algebra and let $\phi_\alpha: A \rightarrow \mathcal{L}(X_\alpha)$ be algebraic representations of A on vector spaces X_α indexed by α . Let $X = \prod_\alpha X_\alpha$ be their direct product. Then the map $\phi: A \rightarrow \mathcal{L}(X)$ defined by

$$\phi(a)((x_\alpha)_\alpha) = (\phi_\alpha(a)(x_\alpha))_\alpha \quad \text{for all } a \in A, (x_\alpha)_\alpha \in X,$$

is an algebraic representation of A on X .

Suppose that A is a lattice-ordered algebra and that each X_α is a vector lattice. Then X is again a vector lattice, with the order defined coordinatewise. If each $\phi_\alpha: A \rightarrow \mathcal{L}_r(X_\alpha)$ is a regular (resp. positive) representation, then it is immediate to check that ϕ is also a regular (resp. positive) representation. If each X_α is order complete and $\phi_\alpha: A \rightarrow \mathcal{L}_r(X_\alpha)$ is a lattice representation, then X is order complete and one can check, using the Riesz–Kantorovich formulas, that $\phi: A \rightarrow \mathcal{L}_r(X)$ is also a lattice representation.

The direct sum $\bigoplus_\alpha X_\alpha$, consisting in the elements of X with finite support, is an A -invariant ideal of X . The associated representation $A \rightarrow \mathcal{L}_r(\bigoplus_\alpha X_\alpha)$ is a regular, positive or lattice representation whenever ϕ is.

Now let X be an arbitrary vector lattice, and let $\{X_\alpha\}_\alpha$ be a family of vector subspaces of X . Then X is said to be the *order direct sum* of $\{X_\alpha\}_\alpha$ if, for every $x \in X_+$, there exist $x_\alpha \in (X_\alpha)_+$, only finitely many of them non-zero, such that $x = \sum_\alpha x_\alpha$, and moreover this decomposition is unique. In this case, the X_α are automatically bands [Zaa97, Theorem 11.3], and X is vector lattice isomorphic to $\bigoplus_\alpha X_\alpha$. When this is the case we will simply denote $X = \bigoplus_\alpha X_\alpha$. As usual, the uniqueness of the decomposition is equivalent to checking that $X_\alpha \cap \sum_{\beta \neq \alpha} X_\beta = 0$ for every α .

5.3 SEMISIMPLE ORDERED MODULES

From this point on, we will assume that all vector lattices are Archimedean.

Recall that, in the purely algebraic context, a non-zero A -module is simple if the only A -invariant subspaces are zero and the whole space. An A -module is semisimple if it is the direct sum of simple A -invariant subspaces. In the ordered context, the following seem to be the natural analogues of these notions.

Definition 5.10. Let A be an algebra, and let X be a vector lattice that is an A -module. We say that X is *simple for A -invariant ideals* if it is non-zero and if it does not have any A -invariant ideal other than 0 and X . We say that X is *semisimple for A -invariant ideals* if it can be expressed as the order direct sum of A -invariant subspaces that are simple for ideals. When there is no risk of confusion about the underlying module structure, we will write *simple for ideals* and *semisimple for ideals*, respectively.

One of the most fundamental results about semisimple modules is that they are characterized by the fact that every A -invariant subspace admits an A -invariant complement. In the ordered context, the following is true.

Theorem 5.11. *Let X be a non-zero vector lattice that is also an A -module. Then the following are equivalent:*

- (i) X is semisimple for ideals,
- (ii) X is the sum of A -invariant ideals that are simple for ideals,
- (iii) Every A -invariant ideal I of X is a projection band and its disjoint complement I^d is A -invariant.

Before presenting the proof some preliminary lemmas are needed. In all these lemmas, X denotes a non-zero vector lattice that is also an A -module. As usual, the A -invariant ideal generated by a set is, by definition, the intersection of all the A -invariant ideals that contain that set.

Lemma 5.12. *Suppose there exists $e \in X_+$ such that the A -invariant ideal it generates is X . Then X has a maximal A -invariant ideal.*

Proof. The proof is a standard maximality argument. Let $\mathcal{M}$ be the family of A -invariant ideals that do not contain e . The zero subspace belongs to $\mathcal{M}$, so that it is non-empty. Order $\mathcal{M}$ by inclusion. Let $\{I_\alpha\}_\alpha$ be a chain in $\mathcal{M}$. It is standard to check that $\bigcup_\alpha I_\alpha$ is again an A -invariant ideal that does not contain e . By Zorn's lemma, $\mathcal{M}$ has a maximal element J . We contend that this element is a maximal A -invariant ideal. Indeed, suppose J' is an A -invariant ideal such that $J \subseteq J' \subseteq X$. If $e \in J'$, then $J' = X$. If $e \notin J'$, then J' belongs to $\mathcal{M}$. Maximality of J implies that in this case $J = J'$. $\square$

Lemma 5.13. *Suppose that every A -invariant ideal I of X is a projection band and that I^d is again A -invariant. Then every non-zero A -invariant ideal of X has a further A -invariant ideal that is simple for ideals.*

Proof. Let $Y \subseteq X$ be a non-zero A -invariant ideal. Without loss of generality, we may assume that it is generated by a single element. It follows from Lemma 5.12 that Y contains a maximal A -invariant ideal $Z \subseteq Y$. By assumption, $Z^d \cap Y$ is again an A -invariant ideal. We claim that it is simple. First note that it is non-zero, for if $Z^d \cap Y = \{0\}$, then $Y \subseteq Z$, contradicting the fact that $Z \subseteq Y$ is proper. Let Z' be an A -invariant ideal contained in $Z^d \cap Y$. The sum $Z + Z'$ is also an A -invariant ideal. The maximality of Z inside Y implies that either $Z + Z' = Z$ or $Z + Z' = Y$; that is, either $Z' = 0$ or $Z' = Z^d \cap Y$. $\square$

Proof of Theorem 5.11. That (i) implies (ii) is obvious. Suppose (ii) holds, that is, suppose that X is the sum of A -invariant ideals that are simple for ideals. Let I be an A -invariant ideal of X . Consider the family

$$\mathcal{I} = \{ J \subseteq X : J \text{ is an } A\text{-invariant ideal and } J \cap I = 0 \}.$$

The family $\mathcal{I}$ is non-empty because it contains the zero ideal. Order it by inclusion. For any given chain in $\mathcal{I}$, its union belongs again to $\mathcal{I}$ and is an upper bound. By Zorn's lemma there exists an element K in $\mathcal{I}$ that is maximal.

We are going to show that X is the order direct sum of I and K . It will then follow from [Zaa97, Theorem 11.3] that I is a projection band and that K , which is A -invariant, is its disjoint complement. Since I and K are ideals, and

$I \cap K = 0$ by construction, we need only check that $I + K = X$; from this and [Zaa97, Theorem 7.6] it will follow that X is the order direct sum of I and K .

Proceed by contradiction: suppose $I + K$ were not X . By assumption, there would exist some A -invariant ideal S simple for ideals such that S is not contained in $I + K$. Since $S \cap (I + K)$ is again an A -invariant ideal contained in S , and it is not S , it must be zero. Then $S + K$ is an A -invariant ideal satisfying $I \cap (S + K) = 0$. Indeed, if $x = s + k \in I$, with $s \in S$ and $k \in K$, then $s = x - k \in S \cap (I + K)$, which implies $s = 0$ and $x = k \in I \cap K$. This, in turn, implies $x = 0$. Thus $S + K \in \mathcal{I}$, which contradicts the maximality of K inside $\mathcal{I}$.

Finally, suppose that (iii) holds. Let $\mathcal{M}$ be the family of collections $\{S_\alpha\}_\alpha$, where every $S_\alpha \subseteq X$ is an A -invariant ideal simple for ideals, satisfying $\sum_\alpha S_\alpha = \bigoplus_\alpha S_\alpha$. Since we are assuming $X \neq 0$, Lemma 5.13 implies that X has an A -invariant ideal that is simple for ideals, so that $\mathcal{M}$ is not empty. Order $\mathcal{M}$ by inclusion. Given a chain in $\mathcal{M}$, let $\{S_\alpha\}$ be all the A -invariant ideals that appear in the collections belonging to the chain. Once we show that $\{S_\alpha\}$ belongs to $\mathcal{M}$, it will follow that it is an upper bound for the chain. Note that, since we are dealing with sums of ideals, every direct sum is an order direct sum (see [Zaa97, Theorem 7.6]). Thus we need only show that the sum of the $\{S_\alpha\}$ is direct. Suppose that $s \in S_\beta$ is expressed as $s = s_1 + \cdots + s_n$, with $s_j \in S_{\beta_j}$ and $\beta_1, \dots, \beta_n$ all different from β . Since $S_\beta, S_{\beta_1}, \dots, S_{\beta_n}$ all belong to a certain family in the chain, their sum must be direct. Hence $s = 0$ and $s_j = 0$. This shows that the sum of $\{S_\alpha\}$ is direct, and therefore that it belongs to $\mathcal{M}$. By Zorn's lemma, $\mathcal{M}$ has a maximal element $(S_\gamma)_\gamma$.

Put $I = \sum_\gamma S_\gamma$. This is again an A -invariant ideal. By assumption, I is a projection band and its disjoint complement I^d is also A -invariant. If $I \neq X$, then by Lemma 5.13, I^d contains an A -invariant ideal S that is simple for ideals. Then the family $\{S_\gamma\} \cup \{S\}$ would violate the maximality of $\{S_\gamma\}$ inside $\mathcal{M}$. Therefore $X = I$ is semisimple. $\square$

Corollary 5.14. *Let X be a vector lattice, let A be an algebra, and suppose that X is an A -module that is semisimple for ideals. Then the A -invariant projection bands and A -invariant ideals of X coincide.*

As a consequence of this theorem we recover a property of Archimedean vector lattices first due to Huijsmans [Hui76].

Corollary 5.15. *Let X be an Archimedean vector lattice. Every ideal in X is a projection band if and only if $X = \bigoplus_{\gamma \in \Gamma} \mathbb{R}$ for some set Γ .*

Proof. Consider X with its natural $\mathbb{R}$ -module structure. With this structure, $\mathbb{R}$ -invariant subspaces are just subspaces, so that now Theorem 5.11 reads as follows: every ideal of X is a projection band if and only if X is semisimple for ideals. The latter is equivalent to saying that $X = \bigoplus_{\gamma \in \Gamma} S_\gamma$, with S_γ projection bands in X that are simple for ideals. But every Archimedean vector lattice of dimension greater than 1 has non-trivial ideals. Hence $\dim S_\gamma = 1$ for every $\gamma \in \Gamma$. $\square$

5.4 ORDERED SCHUR'S LEMMA

The classical Schur's lemma states that the endomorphism algebra of a simple A -module is a division algebra [EH18, Theorem 3.33]. The goal of this section is to give an analogue of Schur's lemma in the ordered context. Before doing so, we need the following fact.

Proposition 5.16. *Let A be a lattice-ordered algebra, let X be a positive A -module and let $S \subseteq X$. Then the A -invariant ideal generated by S is*

$$\left\{ x \in X : |x| \leq \sum_{i=1}^n a_i |s_i| + \sum_{j=1}^m \lambda_j |t_j|, \ n, m \in \mathbb{N}, s_i, t_j \in S, a_i \in A_+, \lambda_j \in \mathbb{R}_+ \right\}.$$

Proof. Denote by I the displayed set. It is direct to verify that I is an ideal. Let us check that it is also A -invariant. Let $x \in I$ and $a \in A$. Using that multiplication by elements of A_+ is a positive map, it follows that

$$|ax| = |a_+x - a_-x| \leq |a_+x| + |a_-x| \leq (a_+ + a_-)|x| = |a||x|.$$

If $|x| \leq \sum_{i=1}^n a_i |s_i| + \sum_{j=1}^m \lambda_j |t_j|$, then

$$|ax| \leq \sum_{i=1}^n (|a|a_i) |s_i| + \sum_{j=1}^m (|a|\lambda_j) |t_j|$$

and therefore $ax \in I$. It is clear from the definition of I that every element of S belongs to I , and that every A -invariant ideal containing S must also contain I . $\square$

Theorem 5.17 (Ordered Schur's lemma). *Let A be a lattice-ordered algebra. Let X and Y be A -modules that are simple for ideals. Let $T: X \rightarrow Y$ be a lattice A -module homomorphism. Then either $T = 0$ or T is injective and $T(X)$ is a majorizing A -invariant sublattice of Y .*

Proof. Being T a lattice and an A -module homomorphism, its kernel $\ker T$ is an A -invariant ideal. Since X is simple for ideals, $\ker T$ must be 0 or X . Suppose $T \neq 0$. Then T is injective and its image $T(X)$ is an A -invariant sublattice of Y . It remains to check that it is majorizing.

Since Y is also simple for ideals, and $T(X) \neq 0$, the smallest A -invariant ideal generated by $T(X)$ in Y is Y . According to Proposition 5.16, for every $z \in Y$ there exist $a_1, \dots, a_n \in A_+$, $\lambda_1, \dots, \lambda_m \in \mathbb{R}_+$ and $x_1, \dots, x_n, y_1, \dots, y_m \in X$ such that

$$\begin{aligned} |z| &\leq \sum_{i=1}^n a_i |T(x_i)| + \sum_{j=1}^m \lambda_j |T(y_j)| \\ &= \sum_{i=1}^n T(a_i |x_i|) + \sum_{j=1}^m T(\lambda_j |y_j|) \\ &= T\left(\sum_{i=1}^n a_i |x_i| + \sum_{j=1}^m \lambda_j |y_j|\right). \end{aligned}$$

This proves that $T(X)$ is majorizing. $\square$

For interval preserving lattice homomorphisms we recover the full strength of Schur's lemma.

Corollary 5.18. *Let A be a lattice-ordered algebra. Let X and Y be A -modules that are simple for ideals. Let $T: X \rightarrow Y$ be a lattice A -module homomorphism that is interval preserving. Then either $T = 0$ or T is a lattice isomorphism.*

Proof. It follows from Theorem 5.17 using that the image of an interval preserving operator is an ideal [DdJ23, Proposition 3.4]. $\square$

Now we can derive some basic properties of A -modules that are semisimple for ideals.

Proposition 5.19. *Let A be a lattice-ordered algebra, and let X and Y be vector lattices that are A -modules. Suppose that X is semisimple for ideals and that there exists a non-zero lattice A -module homomorphism $T: X \rightarrow Y$ that is surjective and interval preserving. Then Y is semisimple for ideals.*

Proof. Since X is semisimple for ideals it can be expressed as $X = \sum_{\alpha} S_{\alpha}$, where the S_{α} are A -invariant ideals of X that are simple for ideals. By Corollary 5.18, each $T(S_{\alpha})$ is lattice A -module isomorphic to S_{α} . In particular, $T(S_{\alpha})$ is also simple for ideals. Moreover, since T is interval preserving, $T(S_{\alpha})$ is an ideal of Y . By Theorem 5.11, $Y = T(X) = \sum_{\alpha} T(S_{\alpha})$ is semisimple for ideals. $\square$

Proposition 5.20. *Let A be a lattice-ordered algebra. Let $\{X_{\alpha}\}$ be a family of vector lattices that are A -modules. Their direct sum $\bigoplus_{\alpha} X_{\alpha}$ is semisimple for ideals if and only if every X_{α} is semisimple for ideals.*

Proof. Suppose first that $X = \bigoplus_{\alpha} X_{\alpha}$ is semisimple for ideals. Let $\pi_{\alpha}: X \rightarrow X_{\alpha}$ be the projection onto the α coordinate. It is direct to check that this map is a non-zero lattice A -module homomorphism that is surjective and interval preserving. It follows from Proposition 5.19 that X_{α} is semisimple for ideals.

Conversely, suppose that every X_{α} is semisimple for ideals; put $X_{\alpha} = \sum_{\beta \in B_{\alpha}} S_{\alpha\beta}$, where each $S_{\alpha\beta}$ is an ideal in X_{α} that is simple for ideals. Let $\iota_{\alpha}: X_{\alpha} \rightarrow X$ be the inclusion sending $x \in X_{\alpha}$ to the element of X that is zero everywhere except in the coordinate α , where it is equal to x . This is a lattice A -module homomorphism that is also interval preserving. In particular, $\iota_{\alpha}(S_{\alpha\beta})$ is again an ideal of X that is simple for ideals. Since

$$X = \sum_{\alpha} \iota_{\alpha}(X_{\alpha}) = \sum_{\alpha} \sum_{\beta \in B_{\alpha}} \iota_{\alpha}(S_{\alpha\beta})$$

we can apply Theorem 5.11 and conclude that X is semisimple for ideals. $\square$

5.5 ORDERED WEDDERBURN–ARTIN THEOREM

An algebra A is said to be semisimple if it is semisimple as an A -module. The Wedderburn–Artin theorem [EH18, Theorem 5.9] states that, for every semisimple algebra A with identity, there exist division algebras $D_1, \dots, D_r$ and integers $n_1, \dots, n_r$ such that A is isomorphic, as an algebra, to

$$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Conversely, any algebra of this form is semisimple.

The goal of this section is to give a version of this result in the case that A is a lattice-ordered algebra. It seems natural that, in this case, the division algebras are also lattice-ordered algebras, so that the $M_{n_i}(D_i)$ have a natural lattice-ordered algebra structure, and that the isomorphism would be a lattice-algebra isomorphism. Note that $M_{n_i}(D_i)$ is semisimple because it can be decomposed into a direct sum of the matrices supported on each of its columns. In the ordered case, this direct sum is an ordered direct sum. Hence, if such an isomorphism exists, then A must be the order direct sum of simple A -invariant subspaces. This is the only additional assumption we need to construct the ordered version of the Wedderburn–Artin theorem.

Definition 5.21. A lattice-ordered algebra A is said to be *order semisimple* if it is the order direct sum of simple A -invariant subspaces.

Before presenting the result we need several preparatory lemmas. First two elementary facts regarding ordered vector spaces and vector lattices.

Lemma 5.22. *Let X be a vector lattice and let Y be an ordered vector space. If there exists a linear map $T: X \rightarrow Y$ that is surjective, positive and order reflecting (i.e., $Tx \geq 0$ implies $x \geq 0$), then Y is a vector lattice isomorphic to X .*

Proof. Since T is an isomorphism of ordered vector spaces, Y is a vector lattice if and only if X is. $\square$

Lemma 5.23. *Let X be a vector lattice and let Y be an ordered vector space. Suppose there exist positive linear maps $i: Y \rightarrow X$ and $p: X \rightarrow Y$ such that $pi = \text{id}_Y$. Then Y is a vector lattice.*

Proof. Let $x, y \in Y$. We claim that their supremum is $p(i(x) \vee i(y))$. Indeed, positivity of p implies

$$p(i(x) \vee i(y)) \geq p(i(x)) = x,$$

and similarly for y . Suppose that $z \geq x, y$. Then positivity of i implies $i(z) \geq i(x) \vee i(y)$. Applying p to this inequality yields $z = p(i(z)) \geq p(i(x) \vee i(y))$. $\square$

For A -modules X and Y , denote by $\text{Hom}_A(X, Y)$ the set of A -module homomorphisms from X to Y . This set is a subspace of $\mathcal{L}(X, Y)$. Furthermore, if X

and Y are vector lattices, then $\text{Hom}_A(X, Y)$ is an ordered vector space with the positive cone inherited from $\mathcal{L}(X, Y)$. When $X = Y$, we denote $\text{Hom}_A(X, X)$ by $\text{End}_A(X)$. Note that $\text{End}_A(X)$ is an ordered algebra.

For every algebra A , denote by A^{op} its opposite algebra. Note that, if A is an ordered or a lattice-ordered algebra, then so is A^{op} .

Lemma 5.24. *Let A be a lattice-ordered algebra with identity. For every $a \in A$, let $R_a: A \rightarrow A$ denote the right multiplication operator. Then $\text{End}_A(A)$ is a lattice-ordered algebra and the map*

$$\begin{aligned} R: A &\longrightarrow \text{End}_A(A)^{\text{op}} \\ a &\longmapsto R_a \end{aligned}$$

is a lattice A -module isomorphism.

Proof. It is standard that R is an A -module isomorphism ([EH18, Lemma 5.4]). By Lemma 5.22, it suffices to show that R is positive and order reflecting. Since the product of positive elements is positive, it is clear that R is positive. On the other hand, suppose that $R_a \geq 0$. Denote by 1_A the identity in A . Then $0 \leq R_a(1_A) = a$. $\square$

The following is immediate to check.

Lemma 5.25. *Let A be an algebra, and let $X_1, \dots, X_n$ be A -modules. The set Λ of matrices where the (i, j) entry is an element of $\text{Hom}_A(X_j, X_i)$ is an algebra with product given by the matrix multiplication. Moreover, if $X_1, \dots, X_n$ are vector lattices, then Λ is an ordered algebra with order defined entrywise.*

Whenever B is an algebra, $M_n(B)$ is again an algebra with the usual matrix operations. In the case that B is an ordered algebra, so is $M_n(B)$ with the order defined entrywise. If, furthermore, B is a lattice-ordered algebra, then so is $M_n(B)$.

Lemma 5.26. *Let A be a lattice-ordered algebra. Suppose that A is order semisimple. Then there exist lattice-ordered division algebras $D_1, \dots, D_r$ and positive integers $n_1, \dots, n_r \in \mathbb{N}$ such that $\text{End}_A(A)$ is lattice-algebra isomorphic to*

$$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Proof. Let $S_1, \dots, S_r$ be simple A -invariant subspaces such that $A = S_1 \oplus \cdots \oplus S_r$, where the sum is an order direct sum. Let $\kappa_i: S_i \rightarrow A$ be the canonical inclusion and let $\pi_i: A \rightarrow S_i$ be the associated band projection.

Let Λ be the ordered algebra introduced in Lemma 5.25 for the A -modules $S_1, \dots, S_r$. Consider the linear map $\Phi: \text{End}_A(A) \rightarrow \Lambda$ defined by $\Phi(\gamma) = (\gamma_{ij})_{i,j}$, where $\gamma_{ij} = \pi_i \circ \gamma \circ \kappa_j$. Since π_i and κ_j are positive, Φ is positive. Moreover, if $\Phi(\gamma)$ is positive (i.e., all γ_{ij} are positive), then so is its preimage

$$\gamma = \sum_{i,j} \kappa_i \circ \gamma_{ij} \circ \pi_j.$$

Hence Φ is order reflecting. By Lemma 5.24 and Lemma 5.22, Λ is a vector lattice and Φ is a lattice isomorphism. It is also an algebra homomorphism: it sends the identity to the identity, and for all $\beta, \gamma \in \text{End}_A(A)$, the (i, j) entry of $\Phi(\beta)\Phi(\gamma)$ is

$$\begin{aligned} \sum_{l=1}^r \beta_{il} \gamma_{lj} &= \sum_{l=1}^r \pi_i \circ \beta \circ \kappa_l \circ \pi_l \circ \gamma \circ \kappa_j \\ &= \pi_i \circ \beta \circ \left(\sum_{l=1}^r \kappa_l \circ \pi_l \right) \circ \gamma \circ \kappa_j \\ &= \pi_i \circ (\beta \circ \gamma) \circ \kappa_j \\ &= (\beta \circ \gamma)_{ij}. \end{aligned}$$

By Schur’s lemma [EH18, Theorem 3.33], $\text{Hom}_A(S_j, S_i)$ is either zero or a division algebra. Group together the S_i that are isomorphic as algebras in the decomposition $A = S_1 \oplus \cdots \oplus S_r$. Then Λ can be expressed as

$$\Lambda = M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $D_1, \dots, D_r$ are ordered division algebras of the form $\text{End}_A(S)$ for some simple A -module S . There is a natural positive inclusion $\iota: D_i \rightarrow M_{n_i}(D_i)$ that sends $x \in D_i$ to the matrix with x in the $(1, 1)$ entry and zero elsewhere, and a corresponding projection $p: D_i \rightarrow M_{n_i}(D_i)$ onto the $(1, 1)$ entry. These extend to an inclusion $\iota: D_i \rightarrow \Lambda$ and a projection $p: \Lambda \rightarrow D_i$ that are still positive. Since Λ is a vector lattice, it follows from Lemma 5.23 that each D_i is a lattice-ordered algebra. $\square$

Theorem 5.27 (Ordered Wedderburn–Artin). *Let A be a lattice-ordered algebra with identity. Suppose that A is order semisimple. Then there exist lattice-ordered division algebras $D_1, \dots, D_r$ and positive integers $n_1, \dots, n_r \in \mathbb{N}$ such that A is lattice-algebra isomorphic to*

$$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r).$$

Proof. By Lemma 5.26, there exist lattice-ordered division algebras $\tilde{D}_1, \dots, \tilde{D}_r$ and positive integers $n_1, \dots, n_r \in \mathbb{N}$ such that $\text{End}_A(A)$ is lattice-algebra isomorphic to

$$M_{n_1}(\tilde{D}_1) \times \cdots \times M_{n_r}(\tilde{D}_r).$$

According to Lemma 5.24, A is lattice-algebra isomorphic to $\text{End}_A(A)^{\text{op}}$. Hence A is lattice-algebra isomorphic to

$$[M_{n_1}(\tilde{D}_1) \times \cdots \times M_{n_r}(\tilde{D}_r)]^{\text{op}} = M_{n_1}(\tilde{D}_1)^{\text{op}} \times \cdots \times M_{n_r}(\tilde{D}_r)^{\text{op}}.$$

It is not difficult to check that transposition defines a lattice-algebra isomorphism between $M_{n_i}(\tilde{D}_i)^{\text{op}}$ and $M_{n_i}(\tilde{D}_i^{\text{op}})$. The result now follows setting $D_i = \tilde{D}_i^{\text{op}}$. $\square$

Using the additional lattice structure, we can now extract some consequences that were not available in the purely algebraic context. These are based on the following fact.

Proposition 5.28. *Let A be a lattice-ordered division algebra. If A is a normed algebra, then it is lattice-algebra isomorphic to $\mathbb{R}$.*

Proof. According to [BD73, Theorem 14.7], every normed division algebra is algebraically isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We will use the following fact: there is no Archimedean order on $\mathbb{C}$ for which the positive cone is generating and closed under multiplication [Wic17c, Proposition 4.1]. This immediately discards the possibility that A is isomorphic to $\mathbb{C}$. Suppose that $\mathbb{H}$ had an Archimedean order for which the positive cone $\mathbb{H}_+$ is generating and closed under multiplication. Then there would exist $a \in \mathbb{R}$ and $v \in \text{span}\{i, j, k\}$, with $v \neq 0$, such that $a + v \in \mathbb{H}_+$ (otherwise, the cone would be contained in $\mathbb{R}$, and therefore would not be generating). Normalizing, we may assume that $\|v\| = 1$. Then $v^2 = -1$ and $C = \text{span}\{1, v\}$ is a subalgebra isomorphic to $\mathbb{C}$. Define $C_+ = C \cap \mathbb{H}_+$. It is clear that C_+ is a cone that induces an Archimedean order on C (namely, the order inherited from $\mathbb{H}$) for which the product of positive elements is positive. We claim that C_+ is also generating. For this, it suffices to check that v can be written as the difference of two elements in C_+ . If $a \geq 0$, then $v = (a + v) - a$ is the difference of two elements of C_+ . If instead $a \leq 0$, then $v \geq -a \geq 0$, and thus $v \in C_+$. The existence of such a positive cone in C contradicts [Wic17c, Proposition 4.1].

Hence A must be isomorphic, as an algebra, to $\mathbb{R}$. There are only two cones in $\mathbb{R}$ that make it into a vector lattice (namely, $[0, \infty)$ and $(-\infty, 0]$), and only in the standard one the product of positive elements is positive. Hence A must be lattice-algebra isomorphic to $\mathbb{R}$ with the usual lattice structure. $\square$

The assumption that A be normed is necessary. In general, there exist Archimedean lattice-ordered division algebras that are not finite-dimensional.

Example 5.29. Consider the real algebra of formal Laurent series:

$$A = \left\{ \sum_{k=-N}^{\infty} a_k x^k : N \in \mathbb{N}, a_k \in \mathbb{R} \right\}.$$

It can be checked that this algebra is a field. It is also an Archimedean vector lattice with order defined by the positive cone

$$A_+ = \left\{ \sum_{k=-N}^{\infty} a_k x^k : N \in \mathbb{N}, a_k \in \mathbb{R}_+ \right\}.$$

It is clear that the product of positive elements is positive. Hence A is an Archimedean lattice-ordered division algebra of infinite dimension.

Corollary 5.30. *Let A be an Archimedean lattice-ordered algebra with identity. Suppose that A is a normed algebra that is order semisimple. Then there exist positive integers $n_1, \dots, n_r$ such that A is lattice-algebra isomorphic to*

$$M_{n_1}(\mathbb{R}) \times \cdots \times M_{n_r}(\mathbb{R}).$$

In particular, A is finite-dimensional.

Proof. By Theorem 5.27, A is lattice-algebra isomorphic to

$$M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for some lattice-ordered division algebras $D_1, \dots, D_r$. Let $\iota_j: D_j \rightarrow M_{n_j}(D_j)$ be the map that sends $x \in D_j$ to the matrix that has x in the $(1, 1)$ entry and zero elsewhere. Extend it to a lattice-algebraic inclusion $\iota_j: D_j \rightarrow A$. Let $\|\cdot\|$ denote the norm in A . Then the map that sends $x \in D_j$ to $\|\iota_j(x)\|$ makes D_j into a normed algebra. It then follows from Proposition 5.28 that $D_1, \dots, D_r$ are all lattice-algebra isomorphic to $\mathbb{R}$. $\square$

For the family of order semisimple normed lattice-ordered algebras with identity, the representation problem turns out to have a positive answer.

Corollary 5.31. *Let A be a normed lattice-ordered algebra with identity. Suppose that A is order semisimple. Then there exists a finite-dimensional vector lattice X and an injective lattice-algebra homomorphism $A \rightarrow \mathcal{L}_r(X)$.*

Proof. According to Corollary 5.30, A is lattice-algebra isomorphic to $M_{n_1}(\mathbb{R}) \times \cdots \times M_{n_r}(\mathbb{R})$ for some $n_1, \dots, n_r \in \mathbb{N}$. This lattice-ordered algebra can be embedded as blocks in the diagonal of $M_n(\mathbb{R})$, where $n = n_1 + \cdots + n_r$. $\square$

6 AM-ALGEBRAS

David Muñoz Lahoz and Pedro Tradacete, *Banach lattice AM-algebras*, Proc. Amer. Math. Soc. **153** (2025), no. 6, 2565–2577.
MR [4892628](#)

The previous chapter studied representations of a Banach lattice algebra A as operators: lattice-algebra homomorphisms $\phi: A \rightarrow \mathcal{L}_r(X)$, together with their weaker variants, into the algebra of regular operators on an order complete Banach lattice. In this chapter we take a different viewpoint, replacing operators by continuous functions. Rather than asking how the elements of A act on some order complete Banach lattice, we ask when A can itself be identified with a space of continuous real-valued functions on some compact Hausdorff space; equivalently, when A is lattice and algebra isometric to a closed sublattice-algebra of $C(K)$.

The template is the space $C(K)$ of real-valued continuous functions on a compact Hausdorff space K , with its uniform norm $\|\cdot\|_\infty$, pointwise order and pointwise operations. The Banach lattice and Banach algebra structures of $C(K)$ are very nicely interwoven: closed lattice ideals and closed algebraic ideals coincide (and are in correspondence with the closed subsets of K as zero sets); closed subalgebras of $C(K)$ are sublattices, while closed sublattices containing the constant functions are automatically subalgebras (a fact behind some proofs of the classical Stone–Weierstrass theorem); and multiplication by positive elements preserves disjointness (in fact, the center of $C(K)$ is precisely the set of multiplication operators).

From the point of view of Banach lattices alone, $C(K)$ -spaces are very well understood. Recall that a Banach lattice X is an AM-space if $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in X_+$, and an AM-space with unit e when there exists a strong unit $e \in X_+$ such that $\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda e\}$ for all $x \in X$. Kakutani proved that AM-spaces are exactly the closed sublattices of $C(K)$ (Theorem 1.2). The non-trivial implication in Kakutani’s theorem is particularly relevant in Banach lattice theory, as it allows us to view elements in abstract Banach lattices as true functions on some compact Hausdorff space, as seen in Section 1.1.3.

From a Banach lattice algebra point of view, it is natural to wonder whether an analogous intrinsic characterization exists for those Banach lattice algebras A which are, lattice and algebra isometrically, closed sublattice-algebras of some $C(K)$. The notion we need for the case with identity has in fact already appeared in this thesis. In Section 3.1, while studying the ideal A_e generated by the identity of a Banach lattice algebra, we introduced an AM-algebra with unit as a Banach lattice algebra which is also an AM-space with unit, the order

unit being at the same time the algebraic identity. There the notion played an auxiliary role — identifying the piece A_e of A with a concrete $C(K)$ -space in Theorem 3.1. Here, AM-algebras with unit return as the main object of study, and we pair them with a counterpart for the case without identity: an AM-algebra with approximate unit is a Banach lattice algebra with a net $(e_\gamma) \subseteq A_+$ that is simultaneously an approximate order unit (i.e., $x^*(e_\gamma) \rightarrow \|x^*\|$ for every positive functional x^*) and an approximate algebraic identity. The main result of this chapter is the following analogue of Kakutani's representation theorem for Banach lattice algebras.

Theorem 6.1.

- (i) *Every AM-algebra with unit is lattice and algebra isometric to $C(K)$ for some compact Hausdorff space K , with the unit corresponding to the constant one function $\mathbb{1}_K$.*
- (ii) *Every AM-algebra with approximate unit is lattice and algebra isometric to a closed sublattice-algebra of $C(K)$ for some compact Hausdorff space K . More precisely, there exists a closed set $F \subseteq K$ such that the AM-algebra is lattice and algebra isometric to the closed sublattice-algebra of $C(K)$:*

$$\{f \in C(K) : f(t) = 0 \text{ for all } t \in F\}.$$

In particular, it embeds as an order and algebraic ideal in $C(K)$.

Let us explain further the analogy between Theorems 1.2 and 6.1. By Kakutani's theorem, every AM-space with unit is lattice isometric to a $C(K)$ space. When we also have a Banach lattice algebra structure, it seems that the most natural compatibility condition between the order unit and the algebraic structure is to ask for the order unit to be also an algebraic identity. According to the previous theorem, this condition characterizes $C(K)$ spaces up to lattice and algebra isometry, just like Kakutani's theorem characterizes $C(K)$ spaces up to lattice isometry as those being an AM-space with unit.

Similarly, by Kakutani's theorem, every AM-space is lattice isometric to a closed sublattice of $C(K)$. Being an AM-space is equivalent to having a positive net (e_γ) such that $x^*(e_\gamma) \rightarrow \|x^*\|$ for every positive functional x^* (see Lemma 6.9). Such a net is what we call an approximate order unit, establishing an analogy with the fact that the norm of every positive functional is attained at an order unit (see Lemma 6.7). When we also have a Banach lattice algebra structure, it seems that the most natural compatibility condition between the AM-space structure and the algebraic structure is to ask for such an approximate order unit to be an approximate algebraic identity. This property characterizes the closed sublattice-algebras of $C(K)$ up to isometry, just like Kakutani's theorem characterizes closed sublattices of $C(K)$, up to isometry, as those having an approximate order unit (i.e., as those being AM-spaces).

In our proof of Theorem 6.1, the following result, which has already made an appearance in Section 3.1, plays a key role.

Proposition 6.2 ([Mar80, Proposition 1.4]). *Let K be a compact Hausdorff space and let $\star: C(K) \times C(K) \rightarrow C(K)$ be a binary operation such that $C(K)$ with pointwise order and product $\star$ is a lattice-ordered algebra. If in addition $\mathbb{1}_K$ is the identity of $\star$, then $\star$ is the pointwise multiplication.*

In other words, if we fix the lattice structure to be given pointwise and the algebraic identity to be given by the constant one function $\mathbb{1}_K$, there exists a unique lattice-ordered algebra structure on $C(K)$. Since this fact is fundamental in our approach, one could wonder whether there might be other Banach lattice algebras with the same property. In Section 6.2, it will be shown that this is not the case. More precisely, we have the following.

Theorem 6.3. *Let A be a Banach lattice algebra with identity e and order $\leq$. Suppose that there is a unique product on A that makes it into a lattice-ordered algebra with identity e and order $\leq$. Then A is lattice and algebra isometric to $C(K)$, with e corresponding to $\mathbb{1}_K$.*

Remark 6.4. Every closed subalgebra of $C(K)$ is automatically a sublattice. We nevertheless speak of closed *sublattice-algebras* of $C(K)$ in Theorem 6.1 and throughout the chapter, to emphasize that the identification preserves both the lattice and the algebra structures, and is not merely an algebraic inclusion.

6.1 CLOSED SUBLATTICE-ALGEBRAS OF $C(K)$

Our goal is to characterize those Banach lattice algebras that are isometric to a closed sublattice-algebra of $C(K)$.

First note that if $f \wedge g = 0$ in $C(K)$, and $h \in C(K)_+$, then $(hf) \wedge g = 0 = (fh) \wedge g$. Any sublattice-algebra of $C(K)$ must also satisfy this condition; that is, any closed sublattice-algebra of $C(K)$ must be a Banach f -algebra. This shows that not every Banach lattice algebra which is also an AM-space must be a sublattice-algebra of some $C(K)$: the Banach lattice ℓ_∞^2 , with product $(x_1, y_1)(x_2, y_2) = (0, x_1x_2)$, is an AM-space, that is also a Banach lattice algebra, but cannot be a sublattice-algebra of $C(K)$, since

$$(1, 0)^2 \wedge (0, 1) = (0, 1) \text{ while } (1, 0) \wedge (0, 1) = (0, 0),$$

(i.e., it is not an f -algebra). Therefore, some relation between the AM-space and algebraic structures has to be required. In the case with identity, this relation is straightforward, and was already introduced in Section 3.1.

Definition 6.5. Let A be a Banach lattice algebra with identity e . If A is also an AM-space with (order) unit e , we say that A is an *AM-algebra with unit e* .

The following theorem asserts that AM-algebras with unit are nothing but algebras of continuous functions, thus proving the first part of Theorem 6.1. This result was already proved as Theorem 3.3 in Section 3.1; we restate it here for convenience.

Theorem 6.6. *Every AM-algebra with unit is algebra and lattice isometric to $C(K)$, for a certain compact Hausdorff space K .*

In order to prove the second part of Theorem 6.1, we need to define the notions of AM-spaces and AM-algebras with approximate unit. The following is a well-known property that motivates our definition.

Lemma 6.7. *A Banach lattice X is an AM-space with unit $e \in X_+$ if and only if $x^*(e) = \|x^*\|$ for all $x^* \in (X^*)_+$.*

Proof. If B_X denotes the unit ball of X and $e \in X_+$ is an order unit, then $B_X = [-e, e]$. For every $x^* \in (X^*)_+$:

$$\|x^*\| = \sup_{x \in (B_X)_+} x^*(x) = \sup_{x \in [0, e]} x^*(x) = x^*(e).$$

Conversely, if $x^* \in (X^*)_+$ and $x \in (B_X)_+$, by assumption

$$0 \leq x^*(x) \leq \|x^*\| = x^*(e).$$

Since this holds for every $x^* \in (X^*)_+$, it follows that $0 \leq x \leq e$. Since $x \in (B_X)_+$ was arbitrary, $(B_X)_+ \subseteq [0, e]$. Also

$$\|e\| = \sup_{x^* \in (B_{X^*})_+} x^*(e) = \sup_{x^* \in (B_{X^*})_+} \|x^*\| = 1,$$

so $[0, e] \subseteq (B_X)_+$. It follows that $B_X = [-e, e]$, and so X is an AM-space with unit e . $\square$

Definition 6.8. Let X be a Banach lattice, and let $(e_\gamma) \subseteq X_+$ be a net. We say that X is an *AM-space with approximate unit* (e_γ) if $x^*(e_\gamma) \rightarrow \|x^*\|$ for every $x^* \in (X^*)_+$. We also say that (e_γ) is an *approximate order unit* of X .

For this definition to make sense, we need to check that spaces with an approximate order unit are indeed AM-spaces. In fact, this definition is equivalent to that of an AM-space.

Lemma 6.9. *A Banach lattice X has an approximate order unit if and only if X is an AM-space.*

Proof. Let $(e_\gamma) \subseteq X_+$ be an approximate order unit. For $x^*, y^* \in (X^*)_+$:

$$\begin{aligned} \|x^* + y^*\| &= \lim_\gamma (x^* + y^*)(e_\gamma) \\ &= \lim_\gamma x^*(e_\gamma) + \lim_\gamma y^*(e_\gamma) \\ &= \|x^*\| + \|y^*\|. \end{aligned}$$

This proves that X^* is an AL-space. Hence X is an AM-space.

Conversely, suppose that X is an AM-space. The positive unit ball $(B_X)_+$ is an increasing net; denote it by (e_γ) . For $x^* \in (X^*)_+$, $x^*(e_\gamma)$ is an increasing and bounded net of real numbers. As such, its limit coincides with its supremum:

$$\lim_\gamma x^*(e_\gamma) = \sup_\gamma x^*(e_\gamma) = \sup_{x \in (B_X)_+} x^*(x) = \|x^*\|.$$

This shows that $(B_X)_+$ is an approximate order unit. $\square$

Remark 6.10. A priori, we do not assume that approximate order units are bounded, see next example. However, if the approximate order unit is countable, then it must be bounded by the Uniform Boundedness Principle.

The previous lemma also shows that, if a Banach lattice has an approximate order unit, then it has a bounded approximate order unit (namely, the positive unit ball). In the separable case, one can even find a countable bounded approximate order unit.

Example 6.11. The following is an example of an approximate order unit in $C[0, 1]$ with no bounded tails. Denote by $M[0, 1]$ the dual of $C[0, 1]$, and by

$$U(\mu_1, \dots, \mu_n; \varepsilon) = \{ f \in C[0, 1] : |\mu_i(f)| < \varepsilon \text{ for } i = 1, \dots, n \},$$

where $\mu_1, \dots, \mu_n \in M[0, 1]$ and $\varepsilon > 0$. These form a basis of neighborhoods at 0 for the weak topology. Define

$$\mathcal{U} = \{ U(\mu_1, \dots, \mu_n; \varepsilon) : \mu_1, \dots, \mu_n \in M[0, 1], \varepsilon > 0 \}.$$

For every $(U, n) \in \mathcal{U} \times \mathbb{N}$, we are going to construct a continuous function $f_{(U, n)} \in U$ with $\|f_{(U, n)}\|_\infty = n$. Suppose $U = U(\mu_1, \dots, \mu_k; \varepsilon)$, and define $\mu = |\mu_1| + \dots + |\mu_k|$. Decompose $\mu = \mu_c + \mu_d$, where μ_c and μ_d are the continuous and discrete parts of μ , respectively. The support of μ_d is at most countable, say $\text{supp}(\mu_d) = \{p_i\}_{i=1}^\infty$. Choose $N \in \mathbb{N}$ such that $\mu_d(\{p_i\}_{i=N}^\infty) < \varepsilon/(2n)$. Since μ_c is continuous, there exists a non-empty open set $V \subseteq [0, 1] \setminus \{p_i\}_{i=1}^N$ such that $\mu_c(V) < \varepsilon/(2n)$. Fix an arbitrary point $x_0 \in V$. By Urysohn's Lemma, there exists a continuous function $f: [0, 1] \rightarrow [0, 1]$ such that $0 \leq f \leq \mathbb{1}_{[0, 1]}$, $\text{supp}(f) \subseteq V$ and $f(x_0) = 1$. Then $f_{(U, n)} = nf$ is such that $\|f_{(U, n)}\|_\infty = n$ and

$$|\mu_i(f_{(U, n)})| \leq n\mu(f) \leq n\mu(V) < \varepsilon \quad \text{for } i = 1, \dots, k$$

so $f_{(U, n)} \in U$.

If we order $\mathcal{U} \times \mathbb{N}$ by setting $(U, n) \leq (V, m)$ if and only if $V \subseteq U$ and $n \leq m$, then the net $\{f_{(U, n)}\}_{(U, n) \in \mathcal{U} \times \mathbb{N}}$ weakly converges to zero and is unbounded. Indeed, if W is a weak neighbourhood of 0, there exists $U_0 \in \mathcal{U}$ such that $U_0 \subseteq W$, and then $f_{(U, n)} \in W$ for every $(U, n) \geq (U_0, 1)$. Yet $f_{(U, n)}$ is unbounded as we increase n ; in particular, it has no bounded tail. It follows that $\{\mathbb{1}_{[0, 1]} + f_{(U, n)}\}_{(U, n) \in \mathcal{U} \times \mathbb{N}}$ is an approximate order unit with no bounded tail.

Of course, AM-spaces with unit are the same as AM-spaces with an approximate order unit that is constant. The following is another characterization of approximate order units that will be useful later.

Lemma 6.12. *Let X be a Banach lattice. A net $(e_\gamma) \subseteq X_+$ is an approximate order unit if and only if X^{**} is lattice isometric to $C(K)$, for a certain compact Hausdorff space K , and $e_\gamma \xrightarrow{w^*} \mathbb{1}_K$.*

Proof. If (e_γ) is an approximate order unit, X is an AM-space by Lemma 6.9, and therefore its bidual, being an AM-space with unit, is lattice isometric to

$C(K)$ for a certain compact Hausdorff space K . Moreover, the element $\mathbb{1}_K$ is such that $\mathbb{1}_K(x^*) = \|x^*\|$ for all $x^* \in (X^*)_+$, because $B_{X^{**}} = [-\mathbb{1}_K, \mathbb{1}_K]$. Then $x^*(e_\gamma) \rightarrow \|x^*\| = \mathbb{1}_K(x^*)$ for all $x^* \in (X^*)_+$; in other words, $e_\gamma \rightarrow \mathbb{1}_K$ in the weak* topology. The converse follows easily from this last observation. $\square$

The algebraic counterpart of approximate order units are approximate algebraic identities. A net (e_γ) in a Banach algebra A is said to be a *left* (resp. *right*) *approximate (algebraic) identity* if $e_\gamma x \rightarrow x$ (resp. $x e_\gamma \rightarrow x$) for all $x \in A$. Of course, this is a standard definition in Banach algebras (see [Pal94, Chapter 5]). When we omit the left or right specification, it means that it is both a left and a right approximate identity.

Putting together the notions of approximate order unit and approximate algebraic identity, we get the approximate analogue of AM-algebras with unit.

Definition 6.13. Let A be a Banach lattice algebra and let $(e_\gamma) \subseteq A_+$. We say that A is an *AM-algebra with approximate unit* (e_γ) if (e_γ) is both an approximate order unit and an approximate algebraic identity.

Example 6.14. (i) An AM-algebra with unit e is also an AM-algebra with approximate unit, in which the approximate unit is the constantly e sequence.

(ii) The Banach lattice c_0 with supremum norm and coordinatewise order and product is an AM-algebra with approximate unit $(\sum_{i=1}^n e_i)_n$, where $e_i(i) = 1$ and $e_i(j) = 0$ for $i \neq j$.

(iii) Any Banach lattice algebra with identity that is not an AM-space shows that we can have approximate algebraic identities without having approximate order units. This is the case for the Banach lattice algebra $\ell_1(\mathbb{Z})$, where the product is given by convolution.

(iv) If X is an AM-space, we can always endow it with the identically zero product, and it will become a Banach lattice algebra with an approximate order unit that has no approximate algebraic identity (unless $X = \{0\}$).

(v) Even if a Banach lattice algebra has an approximate order unit and an approximate algebraic identity, it may not be an AM-algebra with approximate unit, see Example 6.20.

AM-algebras with approximate unit are nothing but the closed sublattices of $C(K)$ that are, at the same time, subalgebras.

Theorem 6.15. *A Banach lattice algebra A is an AM-algebra with approximate unit if and only if it is lattice and algebra isometric to a closed sublattice-algebra of $C(K)$, for a certain compact Hausdorff K .*

For the proof of this theorem, we will need a couple of additional facts about the Arens products. It turns out that identities for the Arens products are closely related to approximate identities in the original algebra.

Proposition 6.16 ([Pal94, Proposition 5.1.9]). *Let A be a Banach algebra, with canonical embedding $j: A \rightarrow A^{**}$, and let (e_γ) be a left (resp. right) approximate identity. If $j(e_\gamma)$ weak* converges to some $e \in A^{**}$, then e is a left (resp. right) identity for the second (resp. first) Arens product.*

Arens regularity of bilinear operators and algebras has been extensively studied in the literature. In this line, G. Buskes and R. Page provided in [BP05] several equivalent conditions for all bilinear and positive operators defined on a Banach lattice to be Arens regular. Here we will only need the following one.

Theorem 6.17 ([BP05]). *Let E be a Banach lattice. Every positive bilinear map $P: E \times E \rightarrow E$ is Arens regular if and only if ℓ_1 does not lattice embed in E .*

We are now ready to prove Theorem 6.15.

Proof of Theorem 6.15. Let A be an AM-algebra with approximate unit $(e_\gamma) \subseteq A_+$. Since A is an AM-space, its bidual A^{**} is lattice isometric to $C(K)$ for a certain compact Hausdorff K . From Theorem 6.17 it follows that A is Arens regular, because, being an AM-space, ℓ_1 does not embed, as a lattice, in A . Endow A^{**} with the Arens product. Then A^{**} is a Banach lattice algebra, and the canonical isometric embedding $j: A \rightarrow A^{**}$ is both a lattice and algebra homomorphism.

By Lemma 6.12, $j(e_\gamma) \xrightarrow{w^*} \mathbb{1}_K$ in $C(K)$. At the same time, by Proposition 6.16, we have that $\mathbb{1}_K$ is the identity of A^{**} , because both Arens products coincide, and $\mathbb{1}_K$ is the w^* -limit of the two-sided approximate identity (e_γ) . Recapitulating, we have that A^{**} is lattice isometric to $C(K)$, and we have endowed this space with a product, the Arens product, that makes it into a Banach lattice algebra with identity $\mathbb{1}_K$. According to Proposition 6.2, this product can only be the pointwise product, so A^{**} is both lattice and algebra isometric to $C(K)$.

Conversely, if A is a closed sublattice-algebra of $C(K)$, then the positive unit ball $(B_A)_+$ is an increasing net that is an approximate order unit by Lemma 6.9. It is also an approximate algebraic identity, by the standard argument of C^* -algebra theory (see [Dav96, Theorem I.4.8] and note that, even though the natural scalar field for C^* -algebras is $\mathbb{C}$, the same argument works for closed subalgebras of a real $C(K)$). Hence A is an AM-algebra with approximate unit. $\square$

At this point, the proof of Theorem 6.1 is almost done. The remaining details are completed after the restatement of the theorem.

Theorem 6.1.

- (i) *Every AM-algebra with unit is lattice and algebra isometric to $C(K)$ for some compact Hausdorff space K , with the unit corresponding to the constant one function $\mathbb{1}_K$.*

(ii) Every AM-algebra with approximate unit is lattice and algebra isometric to a closed sublattice-algebra of $C(K)$ for some compact Hausdorff space K . More precisely, there exists a closed set $F \subseteq K$ such that the AM-algebra is lattice and algebra isometric to the closed sublattice-algebra of $C(K)$:

$$\{f \in C(K) : f(t) = 0 \text{ for all } t \in F\}.$$

In particular, it embeds as an order and algebraic ideal in $C(K)$.

Proof. (i) is contained in Theorem 3.3. To prove (ii), let A be an AM-algebra with approximate unit. According to Theorem 6.15, we can see A as a closed sublattice-algebra of $C(L)$, for a certain compact Hausdorff L . Since A is a closed sublattice of $C(L)$, in virtue of another result by Kakutani (see [Kak41b, Theorem 3]), there exists a family of pairs of points $\{(t_i, s_i)\}_{i \in I} \subseteq L \times L$ and scalars $\{\lambda_i\}_{i \in I} \subseteq [0, 1]$ such that

$$A = \{f \in C(L) : f(t_i) = \lambda_i f(s_i) \text{ for all } i \in I\}.$$

Given $i \in I$, if $f \in A$, then $f^2(t_i) = f(t_i)^2 = \lambda_i^2 f^2(s_i)$, but since also $f^2 \in A$, $f^2(t_i) = \lambda_i f^2(s_i)$, and we must have $\lambda_i^2 f(s_i)^2 = \lambda_i f(s_i)^2$. This only leaves three possibilities: either $f(s_i) = 0$ for every $f \in A$, in which case $f(t_i) = 0 = f(s_i)$, or $\lambda_i = 0$, in which case $f(t_i) = 0$, or $\lambda_i = 1$, in which case $f(t_i) = f(s_i)$.

Collecting these cases, one gets a family of pairs of points $\{(u_j, v_j)\}_{j \in J} \subseteq L \times L$ and a family of points $\{w_p\}_{p \in P} \subseteq L$ such that

$$A = \{f \in C(L) : f(u_j) = f(v_j) \text{ and } f(w_p) = 0 \text{ for all } j \in J, p \in P\}.$$

To obtain the desired result, it only remains to “glue together” u_j and v_j for every $j \in J$. Even though this is a standard procedure, we work out the details for the sake of completeness.

Consider the relation defined by $\{(u_j, v_j)\}_{j \in J}$ in L . Equivalently, $t \sim s$ in L if and only if $f(t) = f(s)$ for all $f \in A$. This is certainly an equivalence relation. Consider the set $K = L / \sim$ with the quotient topology, and let $Q: L \rightarrow K$ be the quotient map. Note that for each $f \in A$, the map $\tilde{f}: K \rightarrow \mathbb{R}$ defined by $\tilde{f}(Qs) = f(s)$ is well-defined and continuous. Whenever $Qs \neq Qt$, there exists $f \in A$ such that $f(s) \neq f(t)$. Hence, $\tilde{f}(Qs) \neq \tilde{f}(Qt)$, which in particular implies that K is Hausdorff. Since $K = Q(L)$, and Q is continuous, it also follows that K is compact.

There is an isometric embedding

$$\begin{array}{ccc} T: & A & \longrightarrow C(K) \\ & f & \longmapsto \tilde{f} \end{array}.$$

Indeed, T is certainly linear and

$$\sup_{Q(s) \in K} |\tilde{f}(Q(s))| = \sup_{s \in L} |f(s)|.$$

Also,

$$|\tilde{f}|(Qs) = |\tilde{f}(Qs)| = |f(s)| = |f|(s)$$

and

$$\widetilde{fg}(Qs) = (fg)(s) = f(s)g(s) = \widetilde{f}(Qs)\widetilde{g}(Qs),$$

for all $f, g \in A$ and $s \in L$, so T is a lattice and algebra homomorphism. Let now F be the closure of $\{Qw_p : p \in P\} \subset K$. It is straightforward to check that

$$T(A) = \{g \in C(K) : g(t) = 0 \text{ for all } t \in F\}. \quad \square$$

Remark 6.18. The previous result could also be compared to other works which study embeddings of an (ordered) Banach algebra as an (ordered) subalgebra of $C(K)$ such as [Ren92]. In [Ren92, Corollary 3.5], H. Render shows that a Banach algebra with a bounded approximate identity and a closed multiplicative cone containing all squares is order and algebra isomorphic to a subalgebra of $C(K)$. Note that this result is quite different from the one presented here, since H. Render is not assuming that the algebra is a lattice, and the conclusion is not of an isometric nature.

Moreover, the condition that squares are positive is closely related to being an *almost f -algebra* (i.e., $a \wedge b = 0$ implies $ab = 0$). This connects with the result by E. Scheffold [Sch81, Satz 2.4] that an almost f -algebra is semi-simple (i.e., its Jacobson radical is zero) if and only if it is isomorphic to a separating sublattice-algebra of $C_0(\Omega)$, where Ω is a locally compact Hausdorff space. Note that this result characterizes separating sublattice-algebras of $C_0(\Omega)$, but only in an isomorphic manner. The theorem given here, however, characterizes sublattice-algebras of $C(K)$ isometrically. Both Render and Scheffold's results use adaptations of the Gelfand transform, while the technique presented here relies heavily on Banach lattice results and Proposition 6.2.

To finish this section, we extract a couple of consequences of the theorem. The first one is straightforward.

Corollary 6.19. *Every AM-algebra with approximate unit is an f -algebra.*

With this corollary we can come back to the example left hanging in Example 6.14.

Example 6.20. Even if a Banach lattice algebra has an approximate order unit and an approximate algebraic identity, it need not be an AM-algebra with approximate unit. Consider the Banach lattice $c_0 \oplus \mathbb{R}$ with supremum norm and coordinatewise order. For $(x, \lambda), (y, \mu) \in c_0 \oplus \mathbb{R}$, define the product:

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu).$$

Then $c_0 \oplus \mathbb{R}$ is a Banach lattice algebra with identity $(0, 1)$. It is also an AM-space, so it has an approximate order unit. However, it is not an AM-algebra with approximate unit, because it is not an f -algebra (if $x \in c_0$ is nonzero, then $(x, 0) \wedge (0, 1) = (0, 0)$, but $(x, 0)(0, 1) = (x, 0) \neq (0, 0)$).

A precise formulation of the second consequence requires working over the complex field. If K is a compact Hausdorff space, the complexification of $C(K)$

may be identified with $C(K, \mathbb{C})$, the space of continuous complex-valued functions on K . Similarly, the complexification $A_{\mathbb{C}}$ of any closed sublattice-algebra A of $C(K)$ (i.e., of any AM-algebra with approximate unit) will be a subalgebra of $C(K, \mathbb{C})$. It is clear that $A_{\mathbb{C}}$, being the complexification of A , is closed under involution. It follows that it is a C^* -algebra, in which the cone of self-adjoint elements with positive spectrum (i.e., the positive elements in the sense of C^* -algebras) is precisely A_+ . Conversely, by a result of S. Sherman [She51], any C^* -algebra $A_{\mathbb{C}}$ in which the cone of positive elements (in the C^* -algebra sense) forms a lattice, must be commutative, hence a closed subalgebra of $C(K, \mathbb{C})$. Moreover, $A_{\mathbb{C}}$ will be the complexification of $A \cap C(K)$, which is a closed sublattice-algebra of $C(K)$, hence an AM-algebra with approximate unit. This discussion is summarized in the final corollary.

Corollary 6.21. *Let A be a Banach lattice algebra. Its complexification $A_{\mathbb{C}}$ can be endowed with a C^* -algebra structure in such a way that A_+ is the cone of self-adjoint elements with positive spectrum if and only if A is an AM-algebra with approximate unit.*

6.2 BANACH LATTICE ALGEBRAS WITH UNIQUE MULTIPLICATION

Recall from Proposition 6.2 that, fixed the pointwise lattice structure and the algebraic identity to be the constant one function, there is a single product in $C(K)$ that makes it a lattice-ordered algebra (namely, the pointwise product). The goal of this section is to prove the converse of this result.

Theorem 6.3. *Let A be a Banach lattice algebra with identity e and order $\leq$. Suppose that there is a unique product on A that makes it into a lattice-ordered algebra with identity e and order $\leq$. Then A is lattice and algebra isometric to $C(K)$, with e corresponding to $\mathbf{1}_K$.*

Remark 6.22. In contrast, given an Archimedean f -algebra with positive identity, there is a unique f -algebra product having the same identity (see [AB06, Theorem 2.58]). Thus, if we only consider f -algebra products, then every single Archimedean f -algebra with positive identity has uniqueness of the product.

Proof of Theorem 6.3. If $A = A_e$, then the result follows from Theorem 3.1. Suppose that $A \neq A_e$. We are going to exhibit another product that makes A a lattice-ordered algebra with identity e .

Let $P: A \rightarrow A$ be the band projection onto A_e , and let P^d be its disjoint complement. Let $\alpha, \beta \in A_e^*$ be multiplicative functionals with $\alpha(e) = \beta(e) = 1$ (if we identify A_e with $C(K)$, α and β are just evaluations at points of K). For $x, y \in A$, define their product $\star$ as

$$x \star y = PxPy + \alpha(Px)P^d y + P^d x \beta(Py).$$

First we have to check that $\star$ is actually a product. It is clear from the definition that $\star$ is bilinear. For convenience of the reader we include the details for

checking associativity: Given $x, y, z \in A$, note that $P(x \star y) = PxPy$ whereas $P^d(x \star y) = \alpha(Px)P^d y + P^d x \beta(Py)$. Then

$$\begin{aligned} (x \star y) \star z &= P(x \star y)Pz + \alpha(P(x \star y))P^d z + P^d(x \star y)\beta(Pz) \\ &= PxPyPz + \alpha(PxPy)P^d z + \alpha(Px)\beta(Pz)P^d y + \beta(Py)\beta(Pz)P^d x, \end{aligned}$$

and

$$\begin{aligned} x \star (y \star z) &= PxP(y \star z) + \alpha(Px)P^d(y \star z) + P^d x \beta(P(y \star z)) \\ &= PxPyPz + \alpha(Px)\alpha(Py)P^d z + \alpha(Px)\beta(Pz)P^d y + \beta(PyPz)P^d x. \end{aligned}$$

Since α and β are multiplicative, it follows that $(x \star y) \star z = x \star (y \star z)$.

Now, it is clear that if $x, y \in A_+$ then $x \star y \in A_+$. Also,

$$x \star e = PxPe + P^d x \beta(Pe) = x \quad \text{and} \quad e \star x = PePx + \alpha(Pe)P^d x = x,$$

so e is still the identity for $\star$. We have thus shown that A , equipped with this product, is a lattice-ordered algebra with identity e .

If $A_e \neq \mathbb{R}$ (i.e., if the compact K has more than one point), different functionals α and β will give different products. Also, if there exist $x, y \in A_e^d$ such that $xy \neq 0$, then $x \star y = 0$, and the product $\star$ is different from the one we started with. Thus, in these cases we have at least two different products.

It remains to check what happens when $A_e = \mathbb{R}$ and $xy = 0$ for all $x, y \in A_e^d$. In this case, any $x, y \in A$ can be written as $x = \lambda e + x'$ and $y = \mu e + y'$ with $\lambda, \mu \in \mathbb{R}$, $x', y' \in A_e^d$, and then

$$xy = \lambda\mu e + \lambda y' + \mu x'.$$

Let $\phi \in (A_e^d)_+^*$ be a non-zero functional, let $x_0 \in (A_e^d)_+$ be a non-zero element, and define the product

$$x \star y = \lambda\mu e + \lambda y' + \mu x' + \phi(x')\phi(y')x_0.$$

This is clearly bilinear. Let $z = \nu e + z'$, with $\nu \in \mathbb{R}$ and $z' \in A_e^d$, then

$$\begin{aligned} (x \star y) \star z &= \lambda\mu\nu e + \lambda\mu z' + \nu(\lambda y' + \mu x' + \phi(x')\phi(y')x_0) \\ &\quad + \phi(\lambda y' + \mu x' + \phi(x')\phi(y')x_0)\phi(z')x_0 \\ &= \lambda\mu\nu e + \lambda\mu z' + \nu\lambda y' + \nu\mu x' \\ &\quad + [\nu\phi(x')\phi(y') + \lambda\phi(y')\phi(z') + \mu\phi(x')\phi(z') + \phi(x')\phi(y')\phi(z')\phi(x_0)]x_0 \\ &= x \star (y \star z), \end{aligned}$$

and this shows that it is associative. If $x \in A_+$, then $\lambda \geq 0$ and $x' \in A_+$, so the product of positive elements is positive. It is straightforward that e is the identity for this product; hence A with $\star$ is a lattice-ordered algebra with identity e . We have chosen ϕ to be a non-zero functional; pick $x \in A_e^d$ such that $\phi(x) \neq 0$. Then $x \star x = \phi(x)^2 x_0 \neq 0$, whereas $x^2 = 0$ in the original product of A . We have again exhibited two different products with the desired properties, and this finishes the proof. $\square$

In view of the results of this chapter it is natural to ask: given an AM-space, is there always a product making it into an AM-algebra? Is the product unique? If not every AM-space admits such a product, can we characterize the AM-spaces that do? In the following chapter we develop the necessary machinery to answer, in particular, these questions.

7 f -ALGEBRA PRODUCTS ON AL AND AM-SPACES

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A. W. Wickstead asked in [Wic17a, Section 4] for a description of the most general Banach lattice algebra structure on AM-spaces and AL-spaces. In this chapter we describe the f -algebra products on these spaces. It will turn out that these products have elegant characterizations; it is not clear at all whether the general situation will also be so tractable.

More precisely, it is shown in Theorem 7.4 that for every AM-space X there exists an associated AM-space W_X such that the f -algebra products on X are in one-to-one correspondence with $(W_X)_+$. Section 7.1 is devoted to the construction of W_X and the proof of Theorem 7.4. This result generalizes the well-known fact that the f -algebra products on $C(K)$, where K is a compact Hausdorff space, are in one-to-one correspondence with $C(K)_+$ (see Proposition 7.1). There also exist results in the literature characterizing all the almost f -algebra products [Sch81] and all the d -algebra products [Bou04] on $C(K)$.

By Kakutani's theorem, every AM-space X embeds lattice isometrically in a space of continuous functions $C(K)$. Since every closed subalgebra of $C(K)$ is also a vector sublattice, a natural stronger question is to characterize which products on X can be embedded lattice and algebra isometrically in a space of continuous functions. By Theorem 7.8, this question turns out to be equivalent to characterizing when X can be isometrically embedded as a closed (order and ring) ideal of $C(K)$.

The products on an AM-space that embed in a space of continuous functions were characterized intrinsically in the previous chapter. In short, these products are the ones for which $(B_X)_+$ is an approximate algebraic identity, where B_X denotes the unit ball of X . It is shown in Section 7.2 that such a product does not always exist but, when it does, it is unique (Corollary 7.7). Following the nomenclature of the previous chapter, we call AM-algebras the AM-spaces that admit such a product. In particular, the AM-algebra product on $C(K)$ is the pointwise product. In Theorem 7.12 it is shown that the AM-algebra product enjoys nice characterizations, similar to those of the pointwise product in a space of continuous functions.

One of the characterizations of an AM-algebra X is that the space W_X is, in some sense, “maximal.” In Section 7.3, we look at the other end and study those AM-spaces for which W_X is minimal. Note that every vector lattice admits an

f -algebra product, namely, the zero product. Thus we are looking at the vector lattices that admit a unique f -algebra product. In Section 7.3, several properties of such spaces are studied. This is motivated by the existence of a simple characterization of AL-spaces that admit only the zero product: an AL-space admits only the zero product if and only if it has no atoms (Corollary 7.18). This is a simple consequence of the characterization of the f -algebra products on AL-spaces that is provided in Theorem 7.17. We are able to show that there exist AM-spaces that only admit the zero product. We are, however, unable to provide a general and simple characterization of such spaces.

7.1 f -ALGEBRA PRODUCTS ON AM-SPACES

The following result characterizing the f -algebra products on AM-spaces with unit follows from [Con74, Theorem 2.2].

Proposition 7.1. *A product $P: C(K) \times C(K) \rightarrow C(K)$ is an f -algebra product if and only if there exists a weight $w \in C(K)_+$ such that*

$$P(f, g)(t) = w(t)f(t)g(t) \quad \text{for all } f, g \in C(K) \text{ and } t \in K. \quad (7.1)$$

This product is submultiplicative if and only if $\|w\|_\infty \leq 1$.

In view of the previous proposition, one could say that the f -algebra products on $C(K)$ are parametrized by $C(K)_+$. The goal of this section is to give a similar characterization for the f -algebra products on general AM-spaces.

Let X be an AM-space. Denote by $\text{hom}(X, \mathbb{R})$ the set of functionals on X that are lattice homomorphisms. The sets

$$K_X = \text{hom}(X, \mathbb{R}) \cap B_{X^*}, K_X^* = K_X \setminus \{0\}, \text{ and } K_X^1 = \text{hom}(X, \mathbb{R}) \cap S_{X^*}$$

are all Hausdorff topological spaces when equipped with the weak* topology. Furthermore, K_X is compact. We say that a real function f defined on K_X is *positively homogeneous* if $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in K_X$ and $\lambda \in \mathbb{R}_+$ such that $\lambda x^* \in K_X$. For instance, the evaluation ev_x at some $x \in X$ defined by

$$\text{ev}_x(x^*) = x^*(x) \text{ for all } x^* \in K_X$$

is a continuous positively homogeneous function on K_X . The set $C_{\text{ph}}(K_X)$ of all continuous positively homogeneous functions on K_X is a vector lattice with respect to the pointwise addition, scalar multiplication, and ordering. Furthermore, the norm of uniform convergence makes $C_{\text{ph}}(K_X)$ an AM-space.

According to a representation result by A. Goulet de Rugy [GdR72, Corollaire 1.31], the map

$$\begin{array}{ccc} X & \longrightarrow & C_{\text{ph}}(K_X) \\ x & \longmapsto & \text{ev}_x \end{array}$$

defines a surjective lattice isometry. From now on, we shall identify X with $C_{\text{ph}}(K_X)$.

Before characterizing all the f -algebra products on X we need to compute its center in terms of K_X . Denote by $C_b^{(0)}(K_X^*)$ the space of continuous functions $f: K_X^* \rightarrow \mathbb{R}$ that are bounded and satisfy $f(\lambda x^*) = f(x^*)$ for all $x^* \in K_X^1$ and $0 < \lambda \leq 1$. Note that this is a closed sublattice-algebra of $C_b(K_X^*)$. In particular, it is an AM-space with unit, the unit being the constant function one.

For every $h \in C_b^{(0)}(K_X^*)$, define the operator $M_h: C_{\text{ph}}(K_X) \rightarrow C_{\text{ph}}(K_X)$ by

$$M_h(f)(x^*) = \begin{cases} h(x^*)f(x^*) & \text{if } x^* \in K_X^* \\ 0 & \text{if } x^* = 0 \end{cases}.$$

Note that $M_h(f)$ is certainly continuous outside 0 and that, if $x_\alpha^* \rightarrow 0$ in K_X with $x_\alpha^* \neq 0$ for all α big enough, then

$$M_h(f)(x_\alpha^*) = h(x_\alpha^*)f(x_\alpha^*) \rightarrow 0$$

since $h(x_\alpha^*)$ is bounded and $f(x_\alpha^*) \rightarrow 0$. This shows that $M_h(f)$ is continuous at 0. Also, $M_h(f)$ is a positively homogeneous function. Hence, the operator M_h is well-defined. Note that it is also central, since it is linear and satisfies $\pm M_h f \leq \|h\|_\infty f$ for all $f \geq 0$. (Side note: every positively homogeneous function must vanish at 0; from now on, we will not write the value at 0 explicitly.)

Next we show that every operator in the center of $C_{\text{ph}}(K_X)$ is of the form described above. We don't claim any originality in this result: it is well-known that central operators in $C_{\text{ph}}(K_X)$ are multiplication operators (see [Wil71, Buc61]). Here, we provide a characterization suited to our purposes using standard techniques.

Proposition 7.2. *Let X be an AM-space. The map*

$$\begin{array}{ccc} M: & C_b^{(0)}(K_X^*) & \longrightarrow \mathcal{Z}(C_{\text{ph}}(K_X)) \\ & h & \longmapsto M_h \end{array}$$

is a lattice-algebra isometry.

Proof. It is direct to check that M is linear and that it is an algebra homomorphism with $M(\mathbb{1}_{K_X^*}) = \text{id}_X$. That it is a lattice homomorphism follows from the fact that $|T|f = |Tf|$ for every $T \in \mathcal{Z}(C_{\text{ph}}(K_X))$ and $f \in C_{\text{ph}}(K_X)_+$ (see [AA02a, Theorem 3.30]). The norm in $\mathcal{Z}(C_{\text{ph}}(K_X))$ is given by $\|T\| = \inf\{\lambda > 0 : \pm T \leq \lambda \text{id}_X\}$ for all $T \in \mathcal{Z}(C_{\text{ph}}(K_X))$ (see [AA02a, Theorem 3.31]). Therefore, once we establish that M is bijective, it will follow that it is an isometry.

That M is injective is straightforward. To show that it is surjective, we will use an adaptation of the arguments in [Buc61, Section 5] (see also [Wic88, Lemma 2.1]). Let $T \in \mathcal{Z}(C_{\text{ph}}(K_X))$ be a central operator. Let $x^* \in K_X^*$ and choose $f, g \in C_{\text{ph}}(K_X)$ with $f(x^*)g(x^*) \neq 0$. Setting

$$k = f - \frac{f(x^*)}{g(x^*)}g \text{ in } C_{\text{ph}}(K_X)$$

we get, as T is central,

$$|(Tk)(x^*)| \leq \|T\| |k(x^*)| = 0.$$

Whence,

$$\frac{(Tf)(x^*)}{f(x^*)} = \frac{(Tg)(x^*)}{g(x^*)}.$$

Accordingly, the function $h: K_X^* \rightarrow \mathbb{R}$ given by

$$h(x^*) = \frac{(Tf)(x^*)}{f(x^*)} \text{ for all } x^* \in K_X^*,$$

where f is chosen arbitrarily in $C_{\text{ph}}(K_X)$ with $f(x^*) \neq 0$, is well-defined. Since $f \in C_{\text{ph}}(K_X)$ is continuous, if $f(x^*) \neq 0$ there exists a weak* open neighbourhood U of x^* in which $f|_U$ does not vanish. Then $h|_U = (Tf)|_U / f|_U$ and, since both f and Tf are continuous, $h|_U$ is continuous. Hence h is locally continuous, and therefore continuous. Also note that h is bounded, since

$$|h(x^*)| = \frac{|Tf(x^*)|}{|f(x^*)|} \leq \|T\| \frac{|f(x^*)|}{|f(x^*)|} = \|T\|.$$

Thus far we have constructed $h \in C_b(K_X^*)$ such that $Tf(x^*) = h(x^*)f(x^*)$ for all $x^* \in K_X^*$ and $f \in C_{\text{ph}}(K_X)$ with $f(x^*) \neq 0$. Note that, if $f(x^*) = 0$, then also $Tf(x^*) = 0$ (because T is central), so the identity $Tf(x^*) = h(x^*)f(x^*)$ still holds in this case. For an element $x^* \in K_X^1$ and $0 < \lambda \leq 1$, using the homogeneity of f and Tf :

$$\lambda h(x^*)f(x^*) = \lambda Tf(x^*) = Tf(\lambda x^*) = h(\lambda x^*)f(\lambda x^*) = \lambda h(\lambda x^*)f(x^*).$$

Choosing an f for which $f(x^*) \neq 0$ shows that $h(x^*) = h(\lambda x^*)$. Hence $h \in C_b^{(0)}(K_X^*)$ and $T = M_h$, as wanted. $\square$

To each function $w: K_X^1 \rightarrow \mathbb{R}$ associate a map $w_{-1}: K_X^* \rightarrow \mathbb{R}$ defined by

$$w_{-1}(x^*) = \frac{w(x^*/\|x^*\|)}{\|x^*\|}, \quad x^* \in K_X^*.$$

Consider

$$W_X = \{ w \in C_b(K_X^1) : w_{-1} \text{ is continuous on } K_X^* \}.$$

We are going to show that W_X is a closed vector sublattice of $C_b(K_X^1)$, and therefore an AM-space. The positive cone of W_X will end up being in one-to-one correspondence with the f -algebra products on X , i.e., $(W_X)_+$ will be our space of weights. Note that we are not asking for w_{-1} to be bounded on K_X^* (in many cases it is not; see Section 7.2), we are only asking for it to be continuous.

From $(\lambda w + w')_{-1} = \lambda w_{-1} + (w')_{-1}$ and $|w|_{-1} = |w_{-1}|$ it follows that W_X is a vector sublattice of $C_b(K_X^1)$. To show that it is closed, let $(w_n) \subseteq W_X$ be a sequence converging to w in $C_b(K_X^1)$. For $\delta > 0$, denote

$$K_X^\delta = \{ x^* \in K_X : \|x^*\| \geq \delta \}.$$

Estimate

$$\begin{aligned} \sup_{x^* \in K_X^\delta} |(w_n)_{-1}(x^*) - w_{-1}(x^*)| &= \sup_{x^* \in K_X^\delta} \frac{|w_n(x^*/\|x^*\|) - w(x^*/\|x^*\|)|}{\|x^*\|} \\ &\leq \frac{\|w_n - w\|_{C_b(K_X^1)}}{\delta}. \end{aligned}$$

This shows that $(w_n)_{-1}$ converges uniformly to w_{-1} on K_X^δ . In particular, w_{-1} is continuous on K_X^δ . Since this is true for every $\delta > 0$, it follows that w_{-1} is continuous on K_X^* . Hence $w \in W_X$ and W_X is a closed vector sublattice of $C_b(K_X^1)$.

Every element of $(W_X)_+$ defines an f -algebra product on X in the following way:

Lemma 7.3. *Let X be an AM-space and let $w \in (W_X)_+$. Put*

$$\begin{aligned} P: C_{\text{ph}}(K_X) \times C_{\text{ph}}(K_X) &\longrightarrow C_{\text{ph}}(K_X) \\ (f, g) &\longmapsto \chi_{K_X^*} w_{-1} f g, \end{aligned}$$

where $\chi_{K_X^*}$ is the characteristic function of K_X^* (so that $P(f, g)$ is extended to vanish at $x^* = 0$). Then P is an f -algebra product on X . If $\|w\|_\infty \leq 1$, then P is submultiplicative.

Proof. The only non-trivial point is to check that, given $f, g \in C_{\text{ph}}(K_X)$, the function $P(f, g)$ is continuous at 0. But this follows readily from the inequalities

$$|P(f, g)(x^*)| = \left| w \left(\frac{x^*}{\|x^*\|} \right) f \left(\frac{x^*}{\|x^*\|} \right) g(x^*) \right| \leq \|w\|_\infty \|f\| \|g(x^*)\|$$

that hold for every $x^* \in K_X^*$. Notice here that the same inequalities yield straightforwardly the submultiplicativity result. $\square$

The main result of this section is that every f -algebra product on X is of this form.

Theorem 7.4. *Let X be an AM-space identified with $C_{\text{ph}}(K_X)$. A product*

$$P: C_{\text{ph}}(K_X) \times C_{\text{ph}}(K_X) \rightarrow C_{\text{ph}}(K_X)$$

is an f -algebra product if and only if there exists a weight $w \in (W_X)_+$ such that

$$P(f, g) = \chi_{K_X^*} w_{-1} f g$$

for all $f, g \in C_{\text{ph}}(K_X)$. This product is submultiplicative if and only if $\|w\|_\infty \leq 1$.

Proof. After the previous lemma, it only remains to show that every f -algebra product is of the desired form and that, if the product is submultiplicative, then $\|w\|_\infty \leq 1$. Let $P: C_{\text{ph}}(K_X) \times C_{\text{ph}}(K_X) \rightarrow C_{\text{ph}}(K_X)$ be an f -algebra product. For every $g \in C_{\text{ph}}(K_X)$, the f -algebra property implies that the map $P(\cdot, g)$ is

an orthomorphism. Thus $P(\cdot, g) \in \mathcal{Z}(C_{\text{ph}}(K_X))$, and by Proposition 7.2 there exists a unique function $h_g \in C_b^{(0)}(K_X^*)$ such that $P(\cdot, g) = M(h_g)$. Consider the map

$$H: \begin{array}{ccc} C_{\text{ph}}(K_X) & \longrightarrow & C_b^{(0)}(K_X^*) \\ g & \longmapsto & h_g \end{array}.$$

That H is a lattice homomorphism follows from a routine calculation involving the injectivity of M and the fact that positive central operators are lattice homomorphisms.

Let $x^* \in K_X^*$ and let $\delta_{x^*} \in C_b^{(0)}(K_X^*)^*$ denote the usual evaluation at x^* functional. The map $H^*\delta_{x^*} = \delta_{x^*} \circ H: C_{\text{ph}}(K_X) \rightarrow \mathbb{R}$ is a lattice homomorphism. Identifying X with $C_{\text{ph}}(K_X)$ (through $x \mapsto \text{ev}_x$), the definition of K_X^* implies that there exist $\phi(x^*) \in K_X^*$ and $w_1(x^*) \in \mathbb{R}_+$ such that $H^*\delta_{x^*} = w_1(x^*)\phi(x^*)$. This means that, for every $x \in X$, $(H^*\delta_{x^*})(\text{ev}_x) = w_1(x^*)\phi(x^*)(x)$. A mere rewrite yields

$$(H \text{ev}_x)(x^*) = w_1(x^*) \text{ev}_x(\phi(x^*)).$$

This way we can define maps $\phi: K_X^* \rightarrow K_X$ and $w_1: K_X^* \rightarrow \mathbb{R}_+$ in the obvious way satisfying $Hg = w_1g \circ \phi$ on K_X^* for every $g \in C_{\text{ph}}(K_X)$. The positive homogeneity of the functions implies that P has the form:

$$P(f, g)(x^*) = w_1(x^*)f(x^*)g(\phi(x^*)) \quad \text{for all } x^* \in K_X^*. \quad (7.2)$$

Next we want to show that, for every $x^* \in K_X^*$ for which $w_1(x^*) \neq 0$, $\phi(x^*)$ is a scalar multiple of x^* . Otherwise, x^* and $\phi(x^*)$ must be disjoint in X^* , because they are non-zero lattice homomorphisms, and therefore atoms in X^* (see [AA02b, Problem 2.3.6]). Accordingly, there exist $x, y \in X_+$ such that

$$\phi(x^*)(x) < x^*(x) \quad \text{and} \quad x^*(y) < \phi(x^*)(y).$$

Then, using that $w_1(x^*) > 0$,

$$\begin{aligned} P(\text{ev}_x, \text{ev}_y)(x^*) &= w_1(x^*)x^*(x)\phi(x^*)(y) \\ &> w_1(x^*)\phi(x^*)(x)x^*(y) = P(\text{ev}_y, \text{ev}_x)(x^*). \end{aligned}$$

This contradicts the fact that every f -algebra product on an Archimedean vector lattice is commutative [Zaa83, Theorem 140.10]. Hence $\phi(x^*)$ must be a positive multiple of x^* whenever $w_1(x^*) \neq 0$.

Define the map $w_2: K_X^* \rightarrow [0, 1]$ by $\phi(x^*) = w_2(x^*)x^*/\|x^*\|$, whenever $w_1(x^*) \neq 0$, and by $w_2(x^*) = 0$ (or any other value) whenever $w_1(x^*) = 0$. Set $w = w_1w_2$. Then (7.2) becomes, using the homogeneity of g ,

$$P(f, g)(x^*) = \frac{w(x^*)}{\|x^*\|} f(x^*)g(x^*).$$

By the positive homogeneity of f and g ,

$$P(f, g)(x^*/\|x^*\|) = \frac{w(x^*/\|x^*\|)}{\|x^*\|^2} f(x^*)g(x^*).$$

Also

$$\frac{1}{\|x^*\|} P(f, g)(x^*) = \frac{w(x^*)}{\|x^*\|^2} f(x^*) g(x^*)$$

and since $P(f, g)$ is positively homogeneous, it follows by choosing appropriate f and g that $w(x^*) = w(x^*/\|x^*\|)$. Hence we can restrict the definition of w to K_X^1 and write

$$P(f, g)(x^*) = \frac{w(x^*/\|x^*\|)}{\|x^*\|} f(x^*) g(x^*) \quad \text{for all } x^* \in K_X^*.$$

It remains to check that $w \in (W_X)_+$. By construction, w is positive. Since P is positive on both coordinates, it must be bounded by some constant $\|P\|$. Consequently, for every $x^* \in K_X^1$ and $f \in C_{\text{ph}}(K_X)$ with $\|f\|_\infty \leq 1$, we have

$$|w(x^*)| f(x^*)^2 = |P(f, f)(x^*)| \leq \|P(f, f)\| \leq \|P\|.$$

Fixed x^* , $f(x^*)$ gets arbitrarily close to 1 as f ranges through $C_{\text{ph}}(K_X)_+$, so $|w(x^*)| \leq \|P\|$. In particular, if P is submultiplicative, then $\|w\|_\infty \leq \|P\| \leq 1$.

To show that w_{-1} is continuous, let $x_\alpha^* \rightarrow x^*$ be a convergent net in K_X^* . Let $f \in C_{\text{ph}}(K_X)$ be such that $f(x^*) \neq 0$. Since both f and $P(f, f)$ are continuous on K_X , we have

$$f(x_\alpha^*)^2 \rightarrow f(x^*)^2 \quad \text{and} \quad w_{-1}(x_\alpha^*) f(x_\alpha^*)^2 \rightarrow w_{-1}(x^*) f(x^*)^2.$$

Choose α_0 such that $|f(x_\alpha^*)^2 - f(x^*)^2| \leq f(x^*)^2/2$ for all $\alpha \geq \alpha_0$. Then

$$\begin{aligned} |w_{-1}(x_\alpha^*) - w_{-1}(x^*)| &\leq \frac{2}{f(x^*)^2} |f(x_\alpha^*)|^2 |w_{-1}(x_\alpha^*) - w_{-1}(x^*)| \\ &= \frac{2}{f(x^*)^2} |f(x_\alpha^*)^2 w_{-1}(x_\alpha^*) - f(x_\alpha^*)^2 w_{-1}(x^*)| \\ &\leq \frac{2}{f(x^*)^2} |f(x_\alpha^*)^2 w_{-1}(x_\alpha^*) - f(x^*)^2 w_{-1}(x^*)| \\ &\quad + \frac{2}{f(x^*)^2} |f(x^*)^2 - f(x_\alpha^*)^2| |w_{-1}(x^*)| \end{aligned}$$

for all $\alpha \geq \alpha_0$. Taking limits in α shows that $w_{-1}(x_\alpha^*) \rightarrow w_{-1}(x^*)$. $\square$

Remark 7.5. Let X be an AM-space with unit, say $X = C(K)$ for a certain compact Hausdorff space K . Then K_X^1 is homeomorphic to K (in particular, it is compact) and therefore $W_X \subseteq C_b(K_X^1) = C(K)$. Since the norm is continuous on K_X (because every positive functional attains its norm at $\mathbf{1}_K$) it follows that $W_X = C(K)$. Hence, for AM-spaces with unit, the previous theorem reduces to Proposition 7.1.

So, for AM-spaces with unit, $W_X = C_b(K_X^1)$, that is, W_X is maximal. The AM-spaces with the property that W_X is maximal are surprisingly well-behaved. They are the object of study of the following section.

7.2 AM-ALGEBRAS REVISITED

According to Kakutani's theorem every AM-space embeds lattice isometrically in a space of continuous functions. It is natural to ask: which of the products in Theorem 7.4 embed lattice-algebra isometrically in a space of continuous functions? (Note that every sublattice-algebra of a space of continuous functions is an f -algebra, so it suffices to consider f -algebra products to characterize *all* products on an AM-space that embed in a space of continuous functions.) The previous chapter answered this question intrinsically: by Theorem 6.15, the AM-spaces that embed lattice and algebra isometrically as a closed sublattice-algebra of some $C(K)$ are precisely the AM-algebras with approximate unit. Recall from Theorem 6.1 that, in fact, the embedding may be chosen so that the AM-algebra appears as an order and algebraic ideal of $C(K)$, of the form

$$\{f \in C(K) : f(t) = 0 \text{ for all } t \in F\}$$

for some closed $F \subseteq K$. In particular, $(B_A)_+$ is then an approximate algebraic identity for A , by the standard C^* -algebra argument (see, for instance, [Bla06, Section II.4]).

In this language, the question of which products in Theorem 7.4 embed in $C(K)$ becomes: which of the submultiplicative products turn the AM-space X into an AM-algebra with approximate unit? Since X is already an AM-space, these products are precisely the ones for which $(B_X)_+$ is an approximate algebraic identity. The following general result shows that there can be at most one such product.

Proposition 7.6. *Let X be a Banach lattice, and let (e_α) be a net in X . Then there exists at most one Banach f -algebra product on X for which (e_α) is an approximate algebraic identity.*

Proof. Let $*$ and $\star$ be two Banach f -algebra products on X . Using that central operators commute, it is not difficult to check that

$$(f * g) \star h = (f \star h) \star g$$

holds for every $f, g, h \in X$. Suppose (e_α) is an approximate algebraic identity for both $*$ and $\star$. Then

$$(f * g) \star e_\alpha = (f \star e_\alpha) \star g.$$

Passing to the limit, this shows that $*$ and $\star$ are equal. □

Corollary 7.7. *Let X be an AM-space. If there exists a product making X into an AM-algebra with approximate unit, then it is unique.*

When such a product exists, we will simply say that X is an *AM-algebra* and call the associated product the *AM-algebra product*. Of course, every AM-space with unit is an AM-algebra, with the AM-algebra product being the pointwise product. More generally, the following theorem characterizes the AM-spaces that are AM-algebras in several ways.

Theorem 7.8. *Let X be an AM-space. The following are equivalent:*

- (i) X is an AM-algebra.
- (ii) $\mathbb{1}_{K_X^1} \in W_X$.
- (iii) The norm is weak* continuous on K_X^* .
- (iv) W_X is maximal (i.e., $W_X = C_b(K_X^1)$).
- (v) X is lattice isometric to a closed subalgebra of $C(K)$, for a certain compact Hausdorff space K .
- (vi) X is lattice isometric to a closed ideal of $C(K)$, for a certain compact Hausdorff space K .
- (vii) X is lattice isometric to $C_0(\Omega)$, for a certain locally compact Hausdorff space Ω .

Proof. Suppose X is an AM-algebra, and let P denote its product. Then P is given, in the terminology of Theorem 7.4, by a weight $w \in (W_X)_+$ with norm $\|w\|_\infty \leq 1$. The unit ball $(f_\alpha) = (B_X)_+$ is an approximate algebraic identity for P , that is, $P(f_\alpha, g) \rightarrow g$ for all $g \in C_{\text{ph}}(K_X)$. This means in particular that

$$0 = \lim_{\alpha} \sup_{y^* \in K_X^1} |g(y^*)| |w(y^*) f_\alpha(y^*) - 1|.$$

Hence $w(y^*) f_\alpha(y^*) \rightarrow 1$ for all $y^* \in K_X^1$. Since $0 \leq f_\alpha(y^*) \leq 1$ for all α , $w(y^*) \geq 1$. On the other hand, since the product is submultiplicative, $w(y^*) \leq 1$. Thus $w = \mathbb{1}_{K_X^1} \in W_X$, and this shows that (i) implies (ii).

If $w = \mathbb{1}_{K_X^1} \in W_X$, then $w_{-1}(x^*) = 1/\|x^*\|$ is continuous on K_X^* , so (ii) implies (iii). If the norm is continuous on K_X^* , then for every $w \in C_b(K_X^1)$ the function

$$w_{-1}(x^*) = \frac{w(x^*/\|x^*\|)}{\|x^*\|}$$

is continuous on K_X^* . This shows that (iii) implies (iv).

To prove that (iv) implies (v), suppose $W_X = C_b(K_X^1)$. In particular, $\mathbb{1}_{K_X^1} \in W_X$, and so

$$P(f, g)(x^*) = \frac{1}{\|x^*\|} f(x^*) g(x^*)$$

defines an f -algebra product on $C_{\text{ph}}(K_X)$. Consider the restriction map

$$\begin{array}{ccc} R: & C_{\text{ph}}(K_X) & \longrightarrow C_b(K_X^1) \\ & f & \longmapsto f|_{K_X^1} \end{array}.$$

Note that this is a lattice isometric embedding. Moreover, it is an algebra homomorphism for the product P in $C_{\text{ph}}(K_X)$ and the pointwise product in $C_b(K_X^1)$. Since $C_b(K_X^1)$ is lattice-algebra isometric to $C(\beta K_X^1)$, where βK_X^1 denotes the Stone-Ćech compactification of K_X^1 , the desired implication follows.

If X is isometric to a closed subalgebra of $C(K)$, then with the product it inherits from $C(K)$ it is an AM-algebra with approximate unit, and therefore it is isometric to a closed ideal of a space of continuous functions (this is part of Theorem 6.1). Hence (v) implies (vi). Suppose X is lattice isometric to $J_F = \{f \in C(K) : f|_F = 0\}$ for some closed $F \subseteq K$. It is then standard to check that the restriction map

$$\begin{aligned} J_F &\longrightarrow C_0(K \setminus F) \\ f &\longmapsto f|_{K \setminus F} \end{aligned}$$

is a (surjective) lattice isometry. This shows that (vi) implies (vii). That (vii) implies (i) is immediate. $\square$

Remark 7.9. In items (v), (vi), and (vii), the AM-algebra product on X is precisely the pointwise product. Indeed, in every case the pointwise product makes X into a lattice-ordered algebra that is lattice isometric to a closed sublattice-algebra of a certain $C(K)$. By uniqueness, this product is the AM-algebra product on X .

Remark 7.10. If X is an AM-algebra, then the AM-algebra product is given by

$$P(f, g)(x^*) = \frac{1}{\|x^*\|} f(x^*)g(x^*) \quad \text{for all } f, g \in C_{\text{ph}}(K_X).$$

In particular, whenever this product is well-defined, the unit ball is an approximate algebraic identity for it.

The following is an example of an AM-space that does not satisfy the conditions of the previous theorem.

Example 7.11. Consider $X = c$ with the usual linear and lattice structures, but with norm

$$\|x\| = (2 \lim_n |x_n|) \vee \|x\|_\infty,$$

where $\|x\|_\infty = \sup_n |x_n|$. It is straightforward to verify that this norm makes X into an AM-space: for $x, y \in X_+$ one has $\lim_n (x_n \vee y_n) = (\lim_n x_n) \vee (\lim_n y_n)$ and $\sup_n (x_n \vee y_n) = (\sup_n x_n) \vee (\sup_n y_n)$, hence $\|x \vee y\| = \|x\| \vee \|y\|$. Consider the point-evaluation functionals δ_j at coordinates $j \in \mathbb{N}$ and the limit functional $\delta_\infty(x) = \lim_n x_n$. These are all lattice homomorphisms. It is easy to check that $\|\delta_j\| = 1$ for every $j \in \mathbb{N}$, whereas $\|\delta_\infty\| = 1/2$. We claim that $w = \mathbb{1}_{K_X^*} \notin W_X$. Indeed, $\delta_j \rightarrow \delta_\infty$ in K_X^* , yet $w_{-1}(\delta_j) = 1$ and

$$w_{-1}(\delta_\infty) = \frac{w(2\delta_\infty)}{\|\delta_\infty\|} = 2.$$

Thus w_{-1} is not continuous on K_X^* .

The following result distinguishes the AM-algebra product among all products in several ways.

Theorem 7.12. *Let X be an AM-algebra and let P be a product on X . The following are equivalent:*

- (i) P is the AM-algebra product on X .
- (ii) $(B_X)_+$ is an approximate algebraic identity for P .
- (iii) When X is identified with $C_{\text{ph}}(K_X)$:

$$P(f, g)(x^*) = \frac{1}{\|x^*\|} f(x^*)g(x^*)$$

for all $f, g \in C_{\text{ph}}(K_X)$ and $x^* \in K_X^*$.

- (iv) For every $z^* \in K_X^1$ and $x, y \in X$:

$$z^*(P(x, y)) = z^*(x)z^*(y).$$

In the case that X is an AM-space with unit e , all of the above are also equivalent to each of the following additional conditions.

- (v) e is the algebraic identity of P .
- (vi) When X is identified with a certain $C(K)$, P is the pointwise product.

Proof. That (i) and (ii) are equivalent is the main content of Theorem 6.1. That (i) implies (iii) is Remark 7.10. If (iii) holds, then

$$P(f, g)(z^*) = f(z^*)g(z^*)$$

for every $z^* \in K_X^1$; this is precisely (iv).

Suppose (iv) holds. Identify X with $C_{\text{ph}}(K_X)$, so that the condition on the product becomes, as before, $P(f, g)(z^*) = f(z^*)g(z^*)$ for every $z^* \in K_X^1$ and $f, g \in C_{\text{ph}}(K_X)$. The restriction map

$$\begin{array}{ccc} R: & C_{\text{ph}}(K_X) & \longrightarrow C_b(K_X^1) \\ & f & \longmapsto f|_{K_X^1} \end{array}$$

is a lattice isometric embedding. The condition on the product implies that it is also an algebra homomorphism. Since $C_b(K_X^1)$ is lattice-algebra isometric to $C(\beta K_X^1)$, (i) follows.

Suppose that X is an AM-space with unit e . Suppose P is the AM-algebra product on X . Since $(B_X)_+$ converges in norm to e , it follows that e is the algebraic identity of P . So (i) implies (v). If e is the algebraic identity of P , then X is an AM-algebra with unit e , and the first item in Theorem 6.1 implies (vi). That (vi) implies (i) is direct (see Remark 7.9). $\square$

The previous two theorems complement very nicely the results of the previous chapter, because they show precisely when an AM-space can be made into an AM-algebra and, in that case, what the AM-algebra product looks like. They also provide practical ways to compute the AM-algebra product in concrete AM-spaces. The following example illustrates how.

Example 7.13. Let X be an AM-algebra and let $\alpha > 0$. A function $f: K_X \rightarrow \mathbb{R}$ is said to be α -homogeneous if $f(\lambda x^*) = \lambda^\alpha f(x^*)$ for all $x^* \in K_X$ and $\lambda \in \mathbb{R}_+$ with $\lambda x^* \in K_X$. The space $C_{ph}^{(\alpha)}(K_X)$ of α -homogeneous continuous functions on K_X is an AM-space. Moreover, it is an AM-algebra with the product given by

$$P(f, g)(x^*) = \frac{1}{\|x^*\|^\alpha} f(x^*)g(x^*) \quad \text{for all } x^* \in K_X \setminus \{0\}.$$

Indeed, since the norm is weak* continuous on K_X^* , P is a well-defined product on $C_{ph}^{(\alpha)}(K_X)$. Moreover, the restriction map

$$\begin{array}{ccc} R: & C_{ph}^{(\alpha)}(K_X) & \longrightarrow C_b(K_X^1) \\ & f & \longmapsto f|_{K_X^1} \end{array}$$

defines a lattice-algebra isometric embedding. Since $C_b(K_X^1)$ is lattice-algebra isometric to the space of continuous functions $C(\beta K_X^1)$, P is the AM-algebra product on X (see Remark 7.9).

The following are some additional tools to check that a product on an AM-space is the AM-algebra product.

Lemma 7.14. *Let X be an AM-algebra.*

- (i) *If Y is a closed subalgebra of X , then Y is also an AM-algebra (in particular, it is automatically a vector sublattice).*
- (ii) *Suppose that $T: X \rightarrow C(K)$ is a lattice isometric embedding and that $T(X)$ is a subalgebra. Then T is an algebra homomorphism.*

Proof.

- (i) Since X embeds isometrically as a subalgebra in some $C(K)$, so does Y . In particular, Y is also a vector sublattice of X , and it is an AM-algebra.
- (ii) The product induced by $C(K)$ in $T(X)$ is precisely the AM-algebra product by Remark 7.9. $\square$

7.3 AL AND AM-SPACES WITH A UNIQUE f -ALGEBRA PRODUCT

We have first described all the f -algebra products on AM-spaces, and then studied a particular product (the AM-algebra product). The existence of this product was characterized by the fact that the space W_X was maximal. We now turn our heads and look at the other end. Can W_X be minimal? In other words: can an AM-space admit a single f -algebra structure? (Namely, the one given by the zero product.) The following result answers this question in a broader context. Recall that a Banach lattice X is said to have *trivial center* if $\mathcal{Z}(X)$ consists only of the scalar multiples of the identity.

Proposition 7.15. *A Banach lattice with trivial center and dimension greater than 1 has a unique f -algebra product.*

Proof. Let X be a Banach lattice with trivial center, and let P be an f -algebra product on X . We are going to show that if P is not the trivial product, then X must be one-dimensional. For every $x \in X_+$, $P(x, \cdot)$ is an orthomorphism, and therefore a central operator. Since the center is trivial, there exists a scalar $\lambda(x) \in \mathbb{R}_+$ such that $P(x, \cdot) = \lambda(x) \text{id}_X$. Let N be the annihilator of the product P . Recall that N is a band. If P is not the zero product, then $N \neq X$, and therefore its disjoint complement N^d is not zero. Let $x, y \in (N^d)_+$. Since $\lambda(x), \lambda(y) \neq 0$ and

$$\lambda(x)y = xy = yx = \lambda(y)x$$

it follows that N^d has dimension one. Hence N^d is a non-trivial projection band. Since the center of X is trivial, it must be $X = N^d$, and thus X is one-dimensional. $\square$

In particular, every AM-space X of dimension greater than 1 with trivial center (see [GdR72] and [Wic88] for examples of such spaces) satisfies $W_X = \{0\}$. The converse, however, is false.

Example 7.16. Let X_1 and X_2 be Banach lattices that only admit the trivial f -algebra product (for example, X_1 and X_2 can be Banach lattices with trivial center and dimension greater than 1). Their direct sum $X = X_1 \oplus_\infty X_2$ has non-trivial center, since both X_1 and X_2 are non-trivial projection bands in X . Let $P: X \times X \rightarrow X$ be an f -algebra product on X . The f -algebra property implies that projection bands are closed under multiplication. Hence the restriction

$$P|_{X_1 \times X_1}: X_1 \times X_1 \rightarrow X_1$$

defines an f -algebra product on X_1 . By assumption, $P|_{X_1 \times X_1} = 0$. Similarly, $P|_{X_2 \times X_2} = 0$. Let $(x_1, x_2), (y_1, y_2) \in X$ and compute

$$\begin{aligned} P((x_1, x_2), (y_1, y_2)) &= P((x_1, 0), (y_1, 0)) + P((x_1, 0), (0, y_2)) \\ &\quad + P((0, x_2), (y_1, 0)) + P((0, x_2), (0, y_2)) \\ &= P((x_1, 0), (0, y_2)) + P((0, x_2), (y_1, 0)) \\ &= 0, \end{aligned}$$

where in the last equality we are using that the product of disjoint elements is 0. Hence $P = 0$.

This prompts a natural question: is there a simple condition characterizing the AM-spaces with a single f -algebra product? As it turns out, for AL-spaces there is such a simple characterization. This characterization will be a simple consequence of the next theorem, which describes all f -algebra products on AL-spaces.

Before presenting the result, let us recall some notions about atoms in Banach lattices (see, for instance, [LZ71, Section 26]). An *atom* in a Banach lattice X is a positive element $a \in X_+$ for which $0 \leq x \leq a$ implies that x is a scalar multiple of a . In this case, the linear span of a is a projection band. Denote by P_a the associated band projection. Thus $P_a x = \lambda_a(x)a$, for a certain $\lambda_a(x) \in \mathbb{R}$.

The map $x \mapsto \lambda_a(x)$ defines a positive linear functional on X , called the *coordinate functional* of a . Let A be a maximal family of norm-one atoms in X , and let $B = B(A)$ denote the band generated by these atoms. When X is order complete, the band projection onto B is given by $Px = \bigvee_{a \in A} \lambda_a(x)a$ for every $x \in X_+$. When X is order continuous,

$$Px = \sum_{a \in A} \lambda_a(x)a = \sup_{F \in \text{Fin}(A)} \sum_{a \in F} \lambda_a(x)a,$$

where $\text{Fin}(A)$ denotes the collection of finite subsets of A .

Theorem 7.17. *Let X be an AL-space and let $\{e_i\}_{i \in I}$ be a maximal family of norm-one atoms in X . Let e_i^* be the coordinate functional of e_i . A product $P: X \times X \rightarrow X$ is an f -algebra product if and only if there exists a weight $w \in \ell_\infty(I)_+$ such that*

$$P(f, g) = \sum_{i \in I} w(i) e_i^*(f) e_i^*(g) e_i \quad \text{for all } f, g \in X.$$

This product is submultiplicative if and only if $\|w\|_\infty \leq 1$.

Proof. Identify X with $L_1(\mu)$, for some measure μ . Let $P: L_1(\mu) \times L_1(\mu) \rightarrow L_1(\mu)$ be an f -algebra product. First assume that μ is σ -finite. Actually, we may assume that μ is a probability measure, since $L_1(\mu)$, with a σ -finite μ , is lattice isometric to $L_1(\mu')$ for some probability measure μ' .

For every $f \in L_1(\mu)$, the operator $P(f, \cdot)$ is in the center of $L_1(\mu)$. By [AA02a, Theorem 3.34], there exists a unique $w(f) \in L_\infty(\mu)$ such that $P(f, g) = w(f)g$ for all $g \in L_1(\mu)$; in fact, $w(f) = P(f, \mathbb{1})$. Thus $P(f, g) = P(f, \mathbb{1})g$ and, repeating the previous argument with $P(\cdot, \mathbb{1})$, it follows that $w(f) = P(f, \mathbb{1}) = P(\mathbb{1}, \mathbb{1})f$. Therefore

$$P(f, g) = P(\mathbb{1}, \mathbb{1})fg \quad \text{for all } f, g \in L_1(\mu). \quad (7.3)$$

The map $w: L_1(\mu) \rightarrow L_\infty(\mu)$, defined by $w(f) = P(\mathbb{1}, \mathbb{1})f$, is a lattice homomorphism. Decompose $L_1(\mu) = L_1(\nu_1) \oplus L_1(\nu_2)$, where ν_1 is a finite atomic measure and ν_2 is a finite non-atomic measure. Both $L_1(\nu_1)$ and $L_1(\nu_2)$ are projection bands in $L_1(\mu)$. Therefore $w|_{L_1(\nu_2)}: L_1(\nu_2) \rightarrow L_\infty(\mu)$ is also a lattice homomorphism. Since the dual $L_\infty(\nu_2)$ of $L_1(\nu_2)$ has no atoms (because ν_2 is non-atomic), $L_1(\nu_2)$ does not have non-zero functionals that are lattice homomorphisms. But $L_\infty(\mu)$ certainly does. So it must be $w|_{L_1(\nu_2)} = 0$, that is, $P(\mathbb{1}, \mathbb{1}) \in L_1(\nu_1)$. After decomposing $f = f_d + f_c, g = g_d + g_c \in L_1(\nu_1) \oplus L_1(\nu_2)$, and using that the pointwise product of disjoint functions is zero, equation (7.3) ends up being

$$P(f, g) = P(\mathbb{1}, \mathbb{1})f_d g_d = P(f_d, g_d).$$

Now suppose that μ is a general measure. For every $f, g \in L_1(\mu)$, the band generated by f and g is isometric to $L_1(\mu')$, where μ' is the restriction of μ to the union of the supports of f and g ; in particular, μ' is σ -finite. Since bands

are subalgebras, $L_1(\mu')$ is also an f -algebra under the inherited product. By the previous argument, $P(f, g) = P(f_d, g_d)$ in $L_1(\mu')$, where f_d and g_d denote the projections of f and g onto the band generated by the atoms in $L_1(\mu')$. Since $L_1(\mu')$ is a band in $L_1(\mu)$, f_d and g_d are also the projections of f and g onto the band generated by the atoms in $L_1(\mu)$.

As in the statement of the theorem, let $\{e_i\}_{i \in I}$ be a maximal family of norm-one atoms in $L_1(\mu)$, and let $\{e_i^*\}_{i \in I}$ be the associated coordinate functionals. If $e_j^*(P(e_i, e_i)) \neq 0$ were true for some $i \neq j$, then

$$P(e_i, e_i) \wedge e_j \geq e_j^*(P(e_i, e_i))e_j \neq 0$$

contradicting the f -algebra property. So $P(e_i, e_i)$ must be a scalar multiple of e_i . Let $w \in \ell_\infty(I)_+$ be defined by $w(i)e_i = P(e_i, e_i)$. Note that w is indeed bounded, because P is positive on both coordinates, and therefore there exists a constant $\|P\|$ satisfying $\|P(f, g)\| \leq \|P\|\|f\|\|g\|$. Using that disjoint elements have product zero, we get

$$\begin{aligned} P(f, g) &= P(f_d, g_d) \\ &= P\left(\sum_{i \in I} e_i^*(f)e_i, \sum_{j \in I} e_j^*(g)e_j\right) \\ &= \sum_{i \in I} e_i^*(f)e_i^*(g)P(e_i, e_i) \\ &= \sum_{i \in I} w(i)e_i^*(f)e_i^*(g)e_i. \end{aligned}$$

Using this expression, it is clear that P is submultiplicative if and only if $\|w\|_\infty \leq 1$. $\square$

Corollary 7.18. *Let X be an AL-space. The only possible f -algebra product on X is the zero product if and only if X has no atoms.*

For a general Banach lattice X , it is true that if X has an atom, then there exists a non-trivial f -algebra product on X . This is proved after the following lemma, which may be of interest by itself.

Lemma 7.19. *Let X be a vector lattice. If a projection band B in X admits a non-trivial f -algebra product, so does X .*

Proof. Suppose B is a projection band in X with a non-trivial f -algebra product P . Let $Q: X \rightarrow B$ be the band projection associated with B , and define $\tilde{P}: X \times X \rightarrow X$ by $\tilde{P}(x, y) = P(Qx, Qy)$. This is certainly a product on X . We will be done once we show that $\tilde{P}$ satisfies the f -algebra property. If $x \wedge y = 0$ in X , then $Qx \wedge Qy = 0$, because Q is a lattice homomorphism. Let $z \in X_+$. Since P is an f -algebra product, $P(Qz, Qx) \wedge Qy = 0$. But also $P(Qz, Qx) \wedge (y - Qy) = 0$, because $P(Qz, Qx) \in B$ and $y - Qy \in B^d$. In conclusion,

$$0 \leq \tilde{P}(z, x) \wedge y \leq P(Qz, Qx) \wedge Qy + P(Qz, Qx) \wedge (y - Qy) = 0,$$

so $\tilde{P}$ does satisfy the f -algebra property. $\square$

Proposition 7.20. *If a vector lattice has an atom, then it has a non-trivial f -algebra product.*

Proof. Let X be a vector lattice and let $a \in X_+$ be an atom. Then $\text{span}\{a\}$ is a projection band that admits a non-trivial f -algebra product (for instance, $P(\lambda a, \mu a) = (\lambda\mu)a$ for all $\lambda, \mu \in \mathbb{R}$). By Lemma 7.19, X admits a non-trivial f -algebra product. $\square$

As we already know, the converse of the previous result is true for AL-spaces, but not in general (it fails already in $C[0, 1]$). In the end, we were not able to find an analogue of Corollary 7.18 for AM-spaces.

Question 7.21. Is there a simple characterization of the AM-spaces X for which the only f -algebra product is the zero product (i.e., for which $W_X = \{0\}$)?

Now that we have dived into the theory of Banach f -algebras, it is natural to extend this theory in the direction of free objects, in analogy with the fruitful development of the theory of free Banach lattices. In Chapter 8 we undertake this extension and construct the free Banach f -algebra generated by a Banach space.

8 FREE BANACH f -ALGEBRAS

David Muñoz-Lahoz and Pedro Tradacete, *Free banach f -algebras*,
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This chapter is devoted to free objects in the category of Banach f -algebras. Given a Banach space E , the free Banach f -algebra generated by E is a Banach f -algebra $\text{FB}fA[E]$ equipped with an isometric embedding $\eta_E: E \rightarrow \text{FB}fA[E]$ through which every contractive operator $E \rightarrow A$, with A an arbitrary Banach f -algebra, factors uniquely as a lattice-algebra homomorphism. The standing template is the theory of free Banach lattices recalled in Section 1.6. The presence of a non-trivial product, however, disallows several arguments central in the lattice-only setting, most notably the positive homogeneity of the expressions in $\text{FVL}[E]$. New techniques are therefore required.

Free objects in the category of Banach lattice algebras had already been considered [dJ21, Wic17a, Wic17b], and abstract arguments from universal algebra yield their existence. To our knowledge, however, no construction in which the lattice, algebraic, and normed structures interact explicitly was known before the present work.

The core of the chapter is contained in Sections 8.3 and 8.4. Starting from the explicit description of the free Archimedean f -algebra $\text{FA}fA[E]$ developed in Section 8.1, we reach $\text{FN}fA[E]$ and $\text{FB}fA[E]$ through the standard procedure of quotienting by the kernel of a maximal seminorm. This procedure does not, in itself, produce an explicit description of the resulting object, because the kernel being quotiented out is not given explicitly. We compute it using a novel structure theorem for Banach f -algebras developed in Section 8.2: the kernel consists of the functions that vanish on the unit ball B_{E^*} (Theorem 8.50)—the same set that is the kernel of the restriction map $\text{FA}fA[E] \rightarrow C(B_{E^*})$. Quotienting yields an injective, contractive lattice-algebra representation of the free normed f -algebra $\text{FN}fA[E]$ inside $C(B_{E^*})$ (Corollary 8.51). It is a curious coincidence: the analogous restriction map for the free vector lattice $\text{FVL}[E]$ is already injective by positive homogeneity, while for $\text{FA}fA[E]$ the failure of injectivity matches exactly what one must quotient out to build a norm.

The structure theorem of Section 8.2 is of independent interest. For every semiprime Banach f -algebra A we construct a contractive representation $A \rightarrow C(K_A)$, where K_A is the compact Hausdorff space of lattice-algebra homomorphisms $A \rightarrow \mathbb{R}$, and characterize it by a natural extension property (Proposition 8.32). Building on this we prove that every real-valued lattice-algebra homomorphism on a closed sublattice-algebra of a Banach f -algebra extends to a lattice-algebra homomorphism on the whole space (Theorem 8.34). The

extension need not preserve the norm; in fact no non-zero Banach f -algebra is λ -injective for any $\lambda \geq 1$ (Theorem 8.37).

The embedding of $\text{FNfA}[E]$ inside $C(B_{E^*})$ has direct structural consequences. The free normed f -algebra generated by a Banach space is always semiprime (Proposition 8.55), and the free Banach f -algebra generated by a finite-dimensional Banach space E admits, up to isomorphism, an explicit description as the space $C([0, 1] \times S_{E^*})$ equipped with pointwise order and the weighted product

$$(f \star g)(r, u) = r f(r, u)g(r, u), \quad (r, u) \in [0, 1] \times S_{E^*}$$

(Theorem 8.56). It follows that $\text{FBfA}[E]$ is semiprime for every finite-dimensional E (Corollary 8.62) and that finite-dimensional Banach spaces of the same dimension generate isomorphic free Banach f -algebras (Corollary 8.64). The latter is subtler than its free-Banach-lattice analogue because $\text{FBfA}[E]$ extends, by definition, only contractive operators: a contractive linear isomorphism between finite-dimensional Banach spaces need not extend to an isomorphism between the free Banach f -algebras it induces (Example 8.93).

The homomorphism $\iota: \text{FNfA}[E] \rightarrow C(B_{E^*})$ extends by continuity to a contractive lattice-algebra homomorphism on $\text{FBfA}[E]$. Is this extension still injective? When it is, we say that $\text{FBfA}[E]$ is *representable in* $C(B_{E^*})$. Section 8.4 reduces representability to a purely algebraic condition: $\text{FBfA}[E]$ is representable in $C(B_{E^*})$ if and only if it is semiprime (Theorem 8.75). This is just a reduction, and not a full answer, because the completion of a semiprime normed f -algebra need not itself be semiprime (Example 8.79). Whether $\text{FBfA}[E]$ is semiprime for every Banach space E remains open; we establish representability at least when E is finite-dimensional (Corollary 8.76) and when $E = L_1(\mu)$ for an arbitrary measure μ .

Section 8.5 examines the lattice and algebraic properties of $\text{FBfA}[E]$, most cleanly under the assumption of representability. Although the free vector lattice $\text{FVL}[E]$ is order dense in $\text{FBL}[E]$ only when E is finite-dimensional, the free normed f -algebra $\text{FNfA}[E]$ is order dense in $\text{FBfA}[E]$ whenever the latter is semiprime (Corollary 8.90). We also track how an operator $T: E \rightarrow F$ behaves under the natural extension $\bar{T}: \text{FBfA}[E] \rightarrow \text{FBfA}[F]$ (Proposition 8.94). For isometric embeddings the picture sharpens (Section 8.5.4): two Banach spaces with smooth duals have lattice-algebra isometric free Banach f -algebras if and only if they are themselves isometric (Theorem 8.98). Section 8.5.5 concerns free Banach f -algebras with identity. Since every Banach f -algebra with a norm-one identity is lattice-algebra isometric to a space of continuous functions, the situation there collapses to a known one: the free Banach f -algebra with identity generated by E is $C(B_{E^*})$.

8.1 FREE ARCHIMEDEAN f -ALGEBRAS

This section is devoted to the construction and study of the following object.

Definition 8.1. Let S be a set. The *free (Archimedean) f -algebra generated by S* is an (Archimedean) f -algebra $\text{FA}fA(S)$ together with a map $\delta: S \rightarrow \text{FA}fA(S)$ such that, for every (Archimedean) f -algebra A and every map $T: S \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FA}fA(S) \rightarrow A$ satisfying $\hat{T} \circ \delta = T$.

The existence of free f -algebras has been long known in the field of universal algebras [Kei95, Section 5.2]: it follows from the fact that f -algebras form an equationable class [Bir61, Theorem VII.8]. Archimedean f -algebras, however, do not form an equationable class [dJ21, Section 8]. Henriksen and Isbell showed in [HI62] that every LLA expression (that is, every expression involving finitely many variables, linear and lattice operations, and a product) that vanishes on the reals, also vanishes on every Archimedean f -algebra. It then follows from a standard argument due to Birkhoff [Bir35] that the free Archimedean f -algebra exists.

In Section 8.1.1 we revisit the result of Henriksen and Isbell through a completely new lens. Their proof and formulation depended heavily on universal algebra language and techniques that may be unfamiliar to the modern analyst. Here we shall use a more operator-centric approach together with a representation theorem.

Admittedly, this may not seem very original. Yet this approach will be used in Section 8.2 to show that a refined version of the result holds for Banach f -algebras. This result will, in turn, be central in the construction of free Banach f -algebras, the main topic of this chapter.

In Sections 8.1.2 and 8.1.3 the lattice and algebraic properties of the free Archimedean f -algebra are explored. This work is the analogue of what Baker [Bak68] and Bleier [Ble73] did for free vector lattices. The reader familiar with free vector lattices will find the properties of free Archimedean f -algebras surprisingly similar and, at the same time, different enough to deserve a separate study.

Even though free f -algebras date back to 1956, several open conjectures regarding free (Archimedean) f -algebras remain (see [Kei95] for a survey on the topic). Some of these have received recent attention, especially the Birkhoff–Pierce conjecture (see for instance [Mad11, Mah07, Wag10, LSS15]). Free f -algebras also play a prominent role in the study of semialgebraic geometry (see [Kei95, Section 5] for an introduction and further references).

8.1.1 Construction of free Archimedean f -algebras

Informally, a lattice-linear-algebraic (LLA) expression is any expression constructed using finitely many variables, the lattice, linear and multiplication operations. An LLA expression is said to vanish on a subset B of an f -algebra if “evaluating” the expression at arbitrary elements of B always gives zero. This section is devoted to proving the following.

Theorem 8.2. *Let Φ be an LLA expression. If Φ vanishes on $\mathbb{R}$, then it also vanishes on every Archimedean f -algebra.*

Once this result is established, it will follow from a standard argument that the free Archimedean f -algebra generated by a set S is the sublattice-algebra of $\mathbb{R}^{\mathbb{R}^S}$ generated by the evaluations δ_s , where $\delta_s(x) = x(s)$ for all $x \in \mathbb{R}^S$ and $s \in S$. But before proceeding to the proof of Theorem 8.2, a more precise definition of LLA expressions is needed.

LLA expressions

Let Ω be a set of formal operations and let T be a set of formal variables. We are going to describe the procedure to construct formal expressions on the variables of T using the operations from Ω . The definition of such expressions is inductive on the complexity of the expression. Define every element of T and the 0-ary operations to be expressions of complexity 1. If $\omega \in \Omega$ is a k -ary operation, for some $k \geq 1$, and $\Phi_1, \dots, \Phi_k$ are expressions of complexity $n - 1$, then $\omega(\Phi_1, \dots, \Phi_k)$ is an expression of complexity n . By construction, every formal expression is a word in the alphabet $T \cup \Omega$ containing only finitely many elements from T . The fact that at most $t_1, \dots, t_n \in T$ appear in Φ is made explicit by writing $\Phi[t_1, \dots, t_n]$. Denote by $\mathcal{E}(T, \Omega)$ the set of formal expressions on the variables of T and using the operations from Ω . The formal operations with words in $\mathcal{E}(T, \Omega)$ make it into an Ω -algebra (see [dJ21, Section 3] for a more detailed introduction to the language of universal algebras).

Let A be an Ω -algebra. Given a map $\phi: T \rightarrow A$ it extends, by induction on complexity, to a unique Ω -algebra homomorphism $\tilde{\phi}: \mathcal{E}(T, \Omega) \rightarrow A$. If $\phi(t) = a_t$, denote $\tilde{\phi}(\Phi[t_1, \dots, t_n])$ by $\Phi(a_{t_1}, \dots, a_{t_n})$.

When Ω consists of a 0-ary operation 0, a unary operation λ for every $\lambda \in \mathbb{R}$, and two binary operations $+$ and $\vee$, the elements of $\mathcal{E}(T, \Omega)$ are called *lattice-linear (LL) expressions* in T . Note that, for this particular Ω , every vector lattice is an Ω -algebra. If Ω contains an additional binary operation $\cdot$, the elements of $\mathcal{E}(T, \Omega)$ are called *lattice-linear-algebraic (LLA) expressions*. In this case, every lattice-ordered algebra is an Ω -algebra. Observe that every LL expression can also be regarded as an LLA expression where $\cdot$ does not appear. Conversely, from every LLA expression $\Phi[t_1, \dots, t_n]$ one can produce an LL expression $\Phi_0[t_1, \dots, t_n]$ in the following way: repeat the same steps as in the inductive construction of Φ except that, every time the product operation is performed, the resulting expression is set to 0. This way, only operations 0, λ , $+$ and $\vee$ appear in Φ_0 , which is therefore a lattice-linear expression. An LLA expression $\Phi[t_1, \dots, t_n]$ is said to *vanish* on a subset B of a lattice-ordered algebra A if $\Phi(a_1, \dots, a_n) = 0$ for every $a_1, \dots, a_n \in B$.

Theorem 8.2 for f -algebras with identity

The proof of Theorem 8.2 is divided into several steps: first, we show the result is true when an identity is present; then we show it is true when the product is identically zero; and finally we prove the general case.

Lemma 8.3. *Let Φ be an LLA expression. If Φ vanishes on $\mathbb{R}$, then it also vanishes on every Archimedean f -algebra with identity.*

Proof. By Theorem 1.30, we may identify A with an f -algebra in $C^\infty(K)$, for a certain compact Hausdorff space K . For $f \in C^\infty(K)$, let $D(f)$ denote the open dense set of points at which f is real-valued. Let $a_1, \dots, a_n \in A$. By definition of the operations in $C^\infty(K)$:

$$\Phi(a_1, \dots, a_n)(t) = \Phi(a_1(t), \dots, a_n(t)) = 0 \quad \text{for every } t \in D(a_1) \cap \dots \cap D(a_n).$$

Since $D(a_1) \cap \dots \cap D(a_n)$ is dense in K , it follows that $\Phi(a_1, \dots, a_n) = 0$. $\square$

Theorem 8.2 for f -algebras with the zero product

It is a well-known fact that every lattice-linear expression vanishing on $\mathbb{R}$ must also vanish on every vector lattice (see, for instance, [Bak68]). This will be key in showing that Theorem 8.2 holds for f -algebras with the zero product. The proof of this fact is done in two steps: first, it is shown that an LLA expression vanishing on $\mathbb{R}_0$ must vanish on every vector lattice with the zero product (Lemma 8.4); then it is shown that every LLA expression vanishing on $\mathbb{R}$ must also vanish on $\mathbb{R}_0$ (Lemma 8.5).

Lemma 8.4. *Let Φ be an LLA expression and let X be a vector lattice. If Φ vanishes on $\mathbb{R}_0$, then it also vanishes on X_0 .*

Proof. At the core of this proof is the fact that $\Phi(x_1, \dots, x_n) = \Phi_0(x_1, \dots, x_n)$ for every $x_1, \dots, x_n \in X_0$. To show this, proceed by induction on k , the complexity of Φ . If $k = 1$, then $\Phi = \Phi_0$ and the result is clear. Suppose it is true for LLA expressions of complexity up to k , and let Φ have complexity $k + 1$. If either $\Phi = \Psi + \lambda\Upsilon$ or $\Phi = \Psi \vee \Upsilon$ for some LLA expressions Ψ and Υ of complexity up to k and $\lambda \in \mathbb{R}$, then $\Phi_0 = \Psi_0 + \lambda\Upsilon_0$ or $\Phi_0 = \Psi_0 \vee \Upsilon_0$, and the desired result follows from applying the induction hypothesis to Ψ and Υ . If instead $\Phi = \Psi\Upsilon$, then by definition $\Phi_0 = 0$ and, since the product in X_0 is identically zero, also

$$\Phi(x_1, \dots, x_n) = \Psi(x_1, \dots, x_n)\Upsilon(x_1, \dots, x_n) = 0.$$

In particular, this applies to $\mathbb{R}_0$. Hence $\Phi_0(\lambda_1, \dots, \lambda_n) = \Phi(\lambda_1, \dots, \lambda_n) = 0$ for every $\lambda_1, \dots, \lambda_n \in \mathbb{R}_0$. Since Φ_0 is a lattice-linear expression that vanishes on $\mathbb{R}$, it must also vanish on every vector lattice. It follows that $0 = \Phi_0(x_1, \dots, x_n) = \Phi(x_1, \dots, x_n)$ for every $x_1, \dots, x_n \in X_0$. $\square$

Lemma 8.5. *Let Φ be an LLA expression. If Φ vanishes on a neighbourhood of 0 in $\mathbb{R}$, then it also vanishes on $\mathbb{R}_0$.*

Proof. Note that $\Phi = \Phi[t_1, \dots, t_n]$ naturally determines a continuous function $\mathbb{R}^n \rightarrow \mathbb{R}$ defined by $(\lambda_1, \dots, \lambda_n) \mapsto \Phi(\lambda_1, \dots, \lambda_n)$. The same is true of Φ_0 . We are going to prove that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\Phi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} = \Phi_0(\lambda_1, \dots, \lambda_n) \quad (8.1)$$

holds for every $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ by induction on the complexity of Φ .

If Φ has complexity 1, then (8.1) is clear. Assume that the result is true if Φ has complexity up to $k - 1$. Let Φ have complexity k . There exist LLA expressions $\Psi[t_1, \dots, t_n]$ and $\Upsilon[t_1, \dots, t_n]$ of complexity up to $k - 1$ such that either:

(i) $\Phi = \lambda\Psi + \Upsilon$ for some $\lambda \in \mathbb{R}$, in which case $\Phi_0 = \lambda\Psi_0 + \Upsilon_0$ and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{\lambda\Psi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n) + \Upsilon(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \\ &= \lambda \lim_{\varepsilon \rightarrow 0^+} \frac{\Psi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} + \lim_{\varepsilon \rightarrow 0^+} \frac{\Upsilon(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \\ &= \lambda\Psi_0(\lambda_1, \dots, \lambda_n) + \Upsilon_0(\lambda_1, \dots, \lambda_n). \end{aligned}$$

(ii) $\Phi = \Psi \vee \Upsilon$, in which case $\Phi_0 = \Psi_0 \vee \Upsilon_0$ and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{\Psi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n) \vee \Upsilon(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \\ &= \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\Psi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \right) \vee \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\Upsilon(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \right) \\ &= \Psi_0(\lambda_1, \dots, \lambda_n) \vee \Upsilon_0(\lambda_1, \dots, \lambda_n). \end{aligned}$$

(iii) $\Phi = \Psi \cdot \Upsilon$, in which case $\Phi_0 = 0$ and

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{\Psi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n) \cdot \Upsilon(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \\ &= \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\Psi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} \right) \left(\lim_{\varepsilon \rightarrow 0^+} \Upsilon(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n) \right) = 0, \end{aligned}$$

where in the last equality we are using that $(\lambda_1, \dots, \lambda_n) \mapsto \Upsilon(\lambda_1, \dots, \lambda_n)$ defines a continuous function on $\mathbb{R}^n$, and that $\Upsilon(0, \dots, 0) = 0$ by construction.

Now suppose there exists a $\delta > 0$ such that $\Phi(\mu_1, \dots, \mu_n) = 0$ for every $(\mu_1, \dots, \mu_n) \in [-\delta, \delta]^n$. Then the limit in (8.1) must vanish, and therefore $\Phi_0(\lambda_1, \dots, \lambda_n) = 0$ for every $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. It was shown in the proof of Lemma 8.4 that, when evaluated in $\mathbb{R}_0$, $\Phi(\lambda_1, \dots, \lambda_n) = \Phi_0(\lambda_1, \dots, \lambda_n) = 0$. Hence Φ vanishes on $\mathbb{R}_0$. $\square$

Equation (8.1) will be of great use in later sections. For convenience, we collect it in a separate lemma and show that the limit is uniform on compact subsets of $\mathbb{R}^n$.

Lemma 8.6. *For every LLA expression $\Phi[t_1, \dots, t_n]$ and $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$:*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\Phi(\varepsilon\lambda_1, \dots, \varepsilon\lambda_n)}{\varepsilon} = \Phi_0(\lambda_1, \dots, \lambda_n).$$

Moreover, the limit is uniform on compact subsets of $\mathbb{R}^n$.

Proof. The proof is based on the following elementary fact: Let K be a compact metric space and let (f_n) and (g_n) be sequences of real-valued continuous functions defined on K such that $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly, for some $f, g \in C(K)$. Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function. Then $h \circ (f_n \times g_n) \rightarrow h \circ (f \times g)$ uniformly, where $f \times g: K \times K \rightarrow \mathbb{R}^2$ is defined by $(f \times g)(k_1, k_2) = (f(k_1), g(k_2))$, $(k_1, k_2) \in K \times K$.

Let us check this. By keeping only a tail of the sequences, we may assume that $\|f_n - f\|_\infty, \|g_n - g\|_\infty < 1$ hold for all $n \in \mathbb{N}$. Let $[a, b]$ be an interval in $\mathbb{R}$ containing the images of f and g . Then $f_n(K), g_n(K) \subseteq [a - 1, b + 1]$ for all $n \in \mathbb{N}$. Being $[a - 1, b + 1]^2$ a compact subset of $\mathbb{R}^2$, the continuous function h is uniformly continuous on it. Fix an arbitrary $\varepsilon > 0$. Let $\delta > 0$ be such that $|h(x, y) - h(x', y')| < \varepsilon$ whenever $|x - x'|, |y - y'| < \delta$ and $x, x', y, y' \in [a - 1, b + 1]$. Let $N \in \mathbb{N}$ be such that $\|f_n - f\|_\infty, \|g_n - g\|_\infty < \delta$ whenever $n \geq N$. It follows that for every $k_1, k_2 \in K$:

$$|h(f_n(k_1), g_n(k_2)) - h(f(k_1), g(k_2))| < \varepsilon$$

whenever $n \geq N$. This proves the fact.

Now let $K \subseteq \mathbb{R}^n$ be a compact subset. We will show that the limit is uniform on K by induction on the complexity of Φ . If $k = 1$, then $\Phi = \Phi_0$ and the limit is uniform on K . Suppose the limit is uniform on K whenever the LLA expression has complexity up to k . Let Φ be an LLA expression of complexity $k + 1$. Then there exist LLA expressions Ψ and Υ of complexity up to k , and a continuous function $h: \mathbb{R}^2 \rightarrow \mathbb{R}$, such that

$$\Phi(\lambda_1, \dots, \lambda_n) = h(\Psi(\lambda_1, \dots, \lambda_n), \Upsilon(\lambda_1, \dots, \lambda_n)).$$

Of course, this continuous function h can only be a linear function, the supremum or the multiplication. Let (r_m) be an arbitrary sequence of positive real numbers decreasing to zero. If h is positively homogeneous (i.e., if it is a linear function or the supremum), then

$$\frac{\Phi(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m} = h\left(\frac{\Psi(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m}, \frac{\Upsilon(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m}\right).$$

By induction hypothesis

$$\left(\frac{\Psi(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m}\right)_m \text{ and } \left(\frac{\Upsilon(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m}\right)_m$$

converge to $\Psi_0(\lambda_1, \dots, \lambda_n)$ and $\Upsilon_0(\lambda_1, \dots, \lambda_n)$, respectively, uniformly on $(\lambda_1, \dots, \lambda_n) \in K$. By the initial observation, the sequence

$$\left(\frac{\Phi(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m}\right)_m$$

also converges to $h(\Psi_0(\lambda_1, \dots, \lambda_n), \Upsilon_0(\lambda_1, \dots, \lambda_n)) = \Phi_0(\lambda_1, \dots, \lambda_n)$ uniformly on $(\lambda_1, \dots, \lambda_n) \in K$.

It remains to check the case when h is the multiplication. In this case, $\Phi_0 = 0$. The sequence

$$(\Upsilon(r_m \lambda_1, \dots, r_m \lambda_n))_m$$

converges to zero uniformly on $(\lambda_1, \dots, \lambda_n) \in K$. Indeed, Υ , when seen as a function defined on $\mathbb{R}^n$, is continuous. This means that, fixed an $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\Upsilon(x_1, \dots, x_n)| < \varepsilon$ whenever $\max_{i=1, \dots, n} |x_i| < \delta$. Let

$$M = \max_{(\lambda_1, \dots, \lambda_n) \in K} |\lambda_i|.$$

Then $|\Upsilon(r_m \lambda_1, \dots, r_m \lambda_n)| < \varepsilon$ as long as $|r_m| < \delta/M$. This proves the claim.

In the expression

$$\frac{\Phi(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m} = \frac{\Psi(r_m \lambda_1, \dots, r_m \lambda_n)}{r_m} \Upsilon(r_m \lambda_1, \dots, r_m \lambda_n)$$

the left term of the product converges to $\Psi_0(\lambda_1, \dots, \lambda_n)$ while the right term converges to 0, both uniformly on K . Again by the initial observation, it follows that the product converges uniformly to 0. $\square$

The desired result now follows directly from the previous lemmas.

Lemma 8.7. *Let Φ be an LLA expression and let X be a vector lattice. If Φ vanishes on a neighbourhood of 0 in $\mathbb{R}$, then it also vanishes on X_0 .*

Theorem 8.2 for Archimedean f -algebras

The only ingredient left to cook Theorem 8.2 is the structural role played by the annihilator $N(A)$, already discussed in Section 1.4: it is both an algebraic ideal and a band, and coincides with the set of nilpotent elements.

Proof of Theorem 8.2. The product in an Archimedean f -algebra A extends to its order completion, making it an Archimedean f -algebra. Since A lattice-algebra embeds in its order completion, it suffices to prove the result for order complete Archimedean f -algebras.

From now on, assume A is order complete. In this case, the annihilator $N(A)$ is a projection band. Let $P: A \rightarrow A$ be the band projection onto $N(A)$, and let P^d be the band projection onto its disjoint complement, $N(A)^d$. The f -algebra condition implies that the product is band preserving, so $P^d x P^d y \in N(A)^d$ for every $x, y \in A$. From the identity

$$xy = (Px + P^d x)(Py + P^d y) = P^d x P^d y$$

follows both that $P(xy) = 0 = PxPy$ and $P^d(xy) = P^d x P^d y$. This shows that P and P^d are lattice-algebra homomorphisms from A onto the f -algebras $N(A)$ and $N(A)^d$, respectively.

Thus, given $a_1, \dots, a_n \in A$,

$$\begin{aligned} \Phi(a_1, \dots, a_n) &= P(\Phi(a_1, \dots, a_n)) + P^d(\Phi(a_1, \dots, a_n)) \\ &= \Phi(Pa_1, \dots, Pa_n) + \Phi(P^d a_1, \dots, P^d a_n), \end{aligned}$$

and we just need to show that $\Phi(Pa_1, \dots, Pa_n) = 0$ in $N(A)$ and $\Phi(P^d a_1, \dots, P^d a_n) = 0$ in $N(A)^d$. Since the product in $N(A)$ is identically zero, the first follows immediately from Lemma 8.7.

It is direct to check that $N(N(A)^d) = \{0\}$, so that the map $L: N(A)^d \rightarrow \text{Orth}(N(A)^d)$ is an injective lattice-algebra homomorphism. But $\text{Orth}(N(A)^d)$ is an Archimedean f -algebra with identity; from Lemma 8.3 it follows that

$$L(\Phi(P^d a_1, \dots, P^d a_n)) = \Phi(LP^d a_1, \dots, LP^d a_n) = 0.$$

Since L is injective, $\Phi(P^d a_1, \dots, P^d a_n) = 0$, and this finishes the proof. $\square$

The free Archimedean f -algebra generated by a set

Now that Theorem 8.2 has been established, we can construct the free Archimedean f -algebra generated by a set S . A word on notation: when we say that a concrete space with a concrete map is *the free object* we mean that they satisfy the required universal property. This is reasonable because any other pair satisfying the universal property will be isomorphic to the given one. Once we have an explicit construction, we will always identify the free object with that particular realization, and consequently denote both of them the same way.

The function space $\mathbb{R}^S$, equipped with pointwise linear operations, order and product, becomes an Archimedean f -algebra. For every $s \in S$, define $\delta_s: \mathbb{R}^S \rightarrow \mathbb{R}$ by $\delta_s(x) = x(s)$ for all $x \in \mathbb{R}^S$. Denote $\text{FAfA}(S) = \text{VLA}\{\delta_s : s \in S\} \subseteq \mathbb{R}^S$ (recall that this means that $\text{FAfA}(S)$ is the smallest sublattice-algebra of $\mathbb{R}^S$ containing every δ_s), and define the map $\delta: S \rightarrow \text{FAfA}(S)$ by $\delta(s) = \delta_s$.

Theorem 8.8. *The free Archimedean f -algebra over a set S is $\text{FAfA}(S)$ together with the map δ .*

Proof. This fact follows from Theorem 8.2 using a standard argument due to Birkhoff [Bir35]. For convenience of the reader, we sketch it here.

Let A be an Archimedean f -algebra, and let $T: S \rightarrow A$ be a map. For every $f \in \text{FAfA}(S)$ there exists an LLA expression $\Phi[t_1, \dots, t_n]$ such that $f = \Phi(\delta_{s_1}, \dots, \delta_{s_n})$ for some $s_1, \dots, s_n \in S$. Define $\hat{T}: \text{FAfA}(S) \rightarrow A$ by $\hat{T}(f) = \Phi(Ts_1, \dots, Ts_n)$. We need to check that $\hat{T}$ is indeed well-defined. Suppose that also $f = \Psi(\delta_{t_1}, \dots, \delta_{t_m})$ for some other LLA expression Ψ and some $t_1, \dots, t_m \in S$. Then for every $x \in \mathbb{R}^S$:

$$\Phi(x(s_1), \dots, x(s_n)) = f(x) = \Psi(x(t_1), \dots, x(t_m)).$$

Since $(x(s_1), \dots, x(s_n), x(t_1), \dots, x(t_m))$ takes all possible values in $\mathbb{R}^{n+m}$ as x ranges through $\mathbb{R}^S$, it follows that the LLA expression $\Phi - \Psi$ vanishes on $\mathbb{R}$. By Theorem 8.2, it must also vanish on A . In particular, $\Phi(Ts_1, \dots, Ts_n) = \Psi(Tt_1, \dots, Tt_m)$. Hence $\hat{T}$ is well-defined, and it is then clear from the definition that it is the unique lattice-algebra homomorphism satisfying $\hat{T}\delta = T$. $\square$

8.1.2 Lattice properties

In this section, we consider the properties of $\text{FAfA}(S)$ as a vector lattice and compare them with those of the free vector lattice $\text{FVL}(S)$.

Before proceeding to the study of such properties, a preliminary result on the simplification of LLA expressions in f -algebras is needed.

Recall that, whenever A is a real algebra, and $B \subseteq A$ is a subset, $\mathbb{R}[B]$ denotes the subalgebra of A generated by B . In other words, $\mathbb{R}[B]$ contains the polynomials in the elements of B with no constant term. Whenever X is a vector lattice, and $Y \subseteq X$ is a subset, $\text{Lat } Y$ denotes the sublattice generated by Y . It is standard that every element $x \in \text{Lat } Y$ can be written as

$$x = \bigvee_{i=1}^n y_i - \bigvee_{j=1}^m z_j,$$

where $y_1, \dots, y_n, z_1, \dots, z_m \in \text{span } Y$.

A d -algebra A is a lattice-ordered algebra in which $a(b \vee c) = (ab) \vee (ac)$ and $(b \vee c)a = (ba) \vee (ca)$ hold for all $b, c \in A$ and $a \in A_+$. Every f -algebra is in particular a d -algebra.

Proposition 8.9. *If A is a d -algebra and $S \subseteq A$ is such that $A = \text{VLA}(S)$, then*

$$A = \text{Lat}(\mathbb{R}[s_+, s_- : s \in S]).$$

Proof. The proof consists in showing that $B = \text{Lat}(\mathbb{R}[s_+, s_- : s \in S])$; it will then clearly follow that $A = B$. Obviously, B is a sublattice, so it is only left to show that B is closed under taking products. Observe that, given $x \in \mathbb{R}[s_+, s_- : s \in S]$, separating its positive and negative coefficients one can write $x = x_p - x_n$, where $x_p, x_n \geq 0$ and $x_p, x_n \in \mathbb{R}[s_+, s_- : s \in S]$. Now let $y \in B$, say

$$y = \bigvee_{i=1}^m u_i - \bigvee_{j=1}^m v_j \quad \text{with } u_i, v_j \in \mathbb{R}[s_+, s_- : s \in S].$$

Using the d -algebra condition:

$$\begin{aligned} xy &= (x_p - x_n) \left(\bigvee_{i=1}^m u_i - \bigvee_{j=1}^m v_j \right) \\ &= x_p \bigvee_{i=1}^m u_i - x_n \bigvee_{i=1}^m u_i - x_p \bigvee_{j=1}^m v_j + x_n \bigvee_{j=1}^m v_j \\ &= \bigvee_{i=1}^m x_p u_i - \bigvee_{i=1}^m x_n u_i - \bigvee_{j=1}^m x_p v_j + \bigvee_{j=1}^m x_n v_j. \end{aligned}$$

Since $x_p u_i, x_p v_j, x_n u_i, x_n v_j \in \mathbb{R}[s_+, s_- : s \in S]$, it follows that $xy \in B$.

Now suppose $x = \bigvee_{k=1}^p w_k$ for some $w_k \in \mathbb{R}[s_+, s_- : s \in S]$. The element x can be rewritten as

$$x = w_{1,p} \vee \bigvee_{k=2}^p (w_k + w_{1,n}) - w_{1,n} = w - w_{1,n},$$

where $w = w_{1,p} \vee \bigvee_{k=2}^p (w_k + w_{1,n}) \in B$ is positive because $w_{1,p} \geq 0$. Then

$$\begin{aligned} xy &= w \bigvee_{i=1}^m u_i - w \bigvee_{j=1}^m v_j - w_{1,n}y \\ &= \bigvee_{i=1}^m wu_i - \bigvee_{j=1}^m wv_j - w_{1,n}y, \end{aligned}$$

where $wu_i, wv_j, w_{1,n}y \in B$ because we have already checked that the product of an element of $\mathbb{R}[s_+, s_- : s \in S]$ by an element of B is in B . Finally, suppose that x is an arbitrary element of B , say $x = \bigvee_{k=1}^p w_k - \bigvee_{l=1}^p z_l$ for some $w_k, z_l \in \mathbb{R}[s_+, s_- : s \in S]$. Then

$$xy = \left(\bigvee_{k=1}^p w_k \right) y - \left(\bigvee_{k=1}^p z_k \right) y \in B$$

because each of the terms is an element of B by the previous case. So B is closed under multiplication, and therefore is a sublattice-algebra. $\square$

Recall that in $\text{FVL}(S)$, δ_s is a weak unit for every $s \in S$, and $\text{FVL}(S)$ has a strong unit if and only if S is finite (in which case $\sup_{s \in S} |\delta_s|$ is a strong unit). In $\text{FAfA}(S)$ the situation is as follows.

Proposition 8.10. *Let S be a non-empty set.*

(i) *The element δ_s is a weak unit of $\text{FAfA}(S)$ for every $s \in S$.*

(ii) *$\text{FAfA}(S)$ does not have a strong unit.*

Proof. (i) Suppose $f \in \text{FAfA}(S)$ were such that $|f| \wedge |\delta_s| = 0$. Write $f = \Phi(\delta_s, \delta_{s_1}, \dots, \delta_{s_n})$, for a certain LLA expression Φ and $s_1, \dots, s_n \in S$. By evaluating the expression $|f| \wedge |\delta_s| = 0$ at $x \in \mathbb{R}^S$ it follows that $0 = f(x) = \Phi(x(s), x(s_1), \dots, x(s_n))$ whenever $x(s) \neq 0$. This implies that the continuous function

$$\begin{array}{ccc} \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R} \\ (\lambda_0, \lambda_1, \dots, \lambda_n) & \longmapsto & \Phi(\lambda_0, \lambda_1, \dots, \lambda_n) \end{array}$$

vanishes as long as $\lambda_0 \neq 0$. Hence it must be identically zero, and therefore $f = 0$ by Theorem 8.2.

- (ii) Suppose $e \in \text{FAfA}(S)$ is a strong unit. The space of real sequences $\mathbb{R}^{\mathbb{N}}$ equipped with pointwise linear operations, order and product is an Archimedean f -algebra. Fix $s_0 \in S$ and let $T: S \rightarrow \mathbb{R}^{\mathbb{N}}$ be defined by $Ts_0 = (n)_{n \in \mathbb{N}}$ and $Ts = 0$ if $s \neq s_0$. Denote $a = Ts_0$. By construction, $\hat{T}(e) \in \text{VLA}\{a\} = \text{Lat}\{\mathbb{R}[a]\}$. More precisely, $\hat{T}(e) \in \text{Lat}\{a, a^2, \dots, a^k\}$, for a certain $k \in \mathbb{N}$. Being finitely generated, $a + a^2 + \dots + a^k$ is a strong unit of $\text{Lat}\{a, a^2, \dots, a^k\}$, so $\hat{T}(e) \leq \lambda(a + a^2 + \dots + a^k)$ for a certain λ . And since $a \leq a^2 \leq \dots \leq a^k$, one arrives at $\hat{T}(e) \leq k\lambda a^k$. By assumption, e is a strong unit, so there exists $\mu > 0$ such that $|\delta_{s_0}|^{k+1} \leq \mu e$. Evaluating $\hat{T}$ at both sides of the inequality yields $a^{k+1} \leq k\lambda\mu a^k$ or, equivalently, $a \leq k\lambda\mu$. This contradicts the fact that a is unbounded. $\square$

If the set S has more than one element, then $\text{FVL}(S)$ does not have non-trivial projection bands. In particular, it does not have atoms and it is not σ -order complete. This is based on a topological argument (see [Ble73]) which also works for $\text{FAfA}(S)$. We sketch it below, after introducing the necessary notation.

Definition 8.11. The *support* of $f \in \mathbb{R}^S$ is defined to be the set

$$\text{supp}(f) = \{x \in S : f(x) \neq 0\}.$$

For a subset $F \subseteq \mathbb{R}^S$ its *support* is defined as

$$\text{supp}(F) = \bigcup_{f \in F} \text{supp}(f).$$

Equip $\mathbb{R}^S$ with the product topology. Then every element of $\text{FAfA}(S)$ is a continuous function and therefore its support is an open set.

Proposition 8.12. *Let S be a set with more than one element. Then $\text{FAfA}(S)$ does not have non-trivial projection bands. In particular, it does not have atoms and it is not σ -order complete.*

Proof. Suppose $\text{FAfA}(S) = B \oplus B^d$, with B a non-trivial projection band. Denote by P the projection onto B , and by P^d its complementary projection. The first step is to show that $\text{supp}(B) \cup \text{supp}(B^d) = S \setminus \{0\}$. Let $x \in S$, $x \neq 0$. Choose $s \in S$ such that $x(s) \neq 0$. Since $0 \neq |\delta_s|(x) = |P\delta_s|(x) + |P^d\delta_s|(x)$, either $|P\delta_s|(x) \neq 0$ or $|P^d\delta_s|(x) \neq 0$. In any case, $x \in \text{supp}(B) \cup \text{supp}(B^d)$. This proves $\text{supp}(B) \cup \text{supp}(B^d) = S \setminus \{0\}$.

The second step is to show that $\text{supp}(B) \cap \text{supp}(B^d) = \emptyset$. If $x \in \text{supp}(B) \cap \text{supp}(B^d)$, there exist positive $f \in B$ and $g \in B^d$ such that $f(x) > 0$ and $g(x) > 0$. Then $(f \wedge g)(x) = f(x) \wedge g(x) > 0$, which contradicts the fact that $f \wedge g \in B \cap B^d = \{0\}$.

We have thus shown that $\text{supp}(B)$ and $\text{supp}(B^d)$ are a disconnection of $S \setminus \{0\}$. This is absurd, since $|S| > 1$. $\square$

When the set S has cardinality one, $\text{FVL}(1)$ can be identified with $\mathbb{R}^2$, and therefore it is both order complete and atomic. For $\text{FAfA}(1)$ the situation is not so straightforward. Recall that $\text{FAfA}(1) = \text{VLA}\{I\} \subseteq \mathbb{R}^{\mathbb{R}}$, where $I(x) = x$ for all $x \in \mathbb{R}$.

Proposition 8.13. (i) *The unique decomposition into non-trivial projection bands of $\text{FAfA}(1)$ is $\text{VLA}\{I_+\} \oplus \text{VLA}\{I_-\}$.*

(ii) *$\text{FAfA}(1)$ has no atoms.*

(iii) *$\text{FAfA}(1)$ is not σ -order complete.*

Proof. (i) Let $B_+ = \text{VLA}\{I_+\}$ and $B_- = \text{VLA}\{I_-\}$. It suffices to check that B_+ and B_- are subspaces with $B_+ \subseteq B_-^d$ satisfying $\text{FAfA}(1) = B_+ \oplus B_-$ as vector spaces (see [MN91, Lemma 1.2.8]). Certainly, B_+ and B_- are subspaces. Since B_+ and B_- have disjoint supports, $B_+ \cap B_- = \{0\}$ and $B_+ \subseteq B_-^d$. According to Proposition 8.9, every $f \in \text{FAfA}(1)$ can be written as

$$f = \bigvee_{i=1}^n [p_i^+(I_+) + p_i^-(I_-)] - \bigvee_{j=1}^m [q_j^+(I_+) + q_j^-(I_-)],$$

where $p_i^+, p_i^-, q_j^+, q_j^-$ are polynomials. Again, since the supports of I_+ and I_- are disjoint, f can be rewritten as

$$f = \left[\bigvee_{i=1}^n p_i^+(I_+) - \bigvee_{j=1}^m q_j^+(I_+) \right] + \left[\bigvee_{i=1}^n p_i^-(I_-) - \bigvee_{j=1}^m q_j^-(I_-) \right].$$

This proves $f \in B_+ + B_-$.

If $\text{FAfA}(1) = B \oplus B^d$, where B is a non-trivial band projection, then, as in Proposition 8.12, $\text{supp}(B)$ and $\text{supp}(B^d)$ form a disconnection of $\mathbb{R} \setminus \{0\}$. So we may assume $\text{supp}(B) = (0, \infty)$. Then $I_+ \in B$ and, since band projections are closed under multiplication, $B_+ \subseteq B$. It follows easily that $B = B_+$, and similarly $B^d = B_-$.

(ii) Let $a > 0$, say $a = \Phi(I)$ for some LLA expression Φ . It is not difficult to show, by induction on the complexity of Φ , that there exists an $m \in \mathbb{N}$ such that

$$\lim_{x \rightarrow 0^+} \frac{\Phi(x)}{x^m} = \infty.$$

On the other hand

$$\lim_{x \rightarrow 0^+} \frac{x^m \wedge \Phi(x)}{x^m} = 1,$$

so it cannot be the case that $(I_+)^m \wedge a$ is a scalar multiple of a . Hence, a is not an atom.

- (iii) Even though the sequence $f_n = (I_+)^n \wedge I_+$ is bounded below by 0, it does not have an infimum. Indeed, suppose f is a positive lower bound. Then $f(t) = 0$ for $t \in [0, 1)$ and, since f is continuous, there must exist $\delta > 0$ such that $0 \leq f(t) \leq 1/2$ for $t \in [1, 1 + \delta]$. Choose $N \in \mathbb{N}$ big enough so as to have

$$1/2 < (1 + \delta)^N - (1 + \delta).$$

By continuity, there exists $0 < \delta' < \delta$ such that

$$1/2 < (1 + \delta')^N - (1 + \delta') < 1.$$

It then follows that $[((I_+)^N - I_+)_+ \wedge I_+] \vee f$ is a lower bound for (f_n) that is strictly bigger than f . Hence (f_n) does not have an infimum. $\square$

Remark 8.14. The free vector lattice $\text{FVL}(S)$ has the *countable chain condition*: a disjoint collection in $\text{FVL}(S)$ is at most countable. The reason is purely topological: the supports of a disjoint collection of functions form a disjoint collection of open subsets of $\mathbb{R}^S$, and such a collection is at most countable (see [RS64]). The very same argument also shows that $\text{FAfA}(S)$ has the countable chain condition.

Remark 8.15. Let T be a subset of S . The free Archimedean f -algebra $\text{FAfA}(T)$ may be viewed as the sublattice-algebra of $\text{FAfA}(S)$ generated by δ_t for $t \in T$. The unique lattice-algebra homomorphism $P: \text{FAfA}(S) \rightarrow \text{FAfA}(S)$ determined by $P\delta_s = \delta_s$ if $s \in T$ and $P\delta_s = 0$ if $s \notin T$, defines a projection onto $\text{FAfA}(T)$.

Similarly, the free vector lattice $\text{FVL}(S)$ may be viewed as the sublattice of $\text{FAfA}(S)$ generated by δ_s for $s \in S$. Equip $\text{FVL}(S)$ with the zero product, which as usual we denote by $\text{FVL}(S)_0$. The map $\delta: S \rightarrow \text{FVL}(S)_0$ extends to a unique lattice-algebra homomorphism $P: \text{FAfA}(S) \rightarrow \text{FVL}(S)_0$. Forgetting about the product on $\text{FVL}(S)_0$, the map

$$P: \text{FAfA}(S) \rightarrow \text{FVL}(S) \subseteq \text{FAfA}(S)$$

is a lattice homomorphic projection. The explicit expression of this projection is

$$(Pf)(x) = \lim_{t \rightarrow 0^+} \frac{f(tx)}{t}, \quad \text{where } f \in \text{FAfA}(S).$$

Indeed, suppose $f = \Phi(\delta_{s_1}, \dots, \delta_{s_n})$ for some LLA expression Φ and $s_1, \dots, s_n \in S$. By Lemma 8.6,

$$\lim_{t \rightarrow 0^+} \frac{f(tx)}{t} = \lim_{t \rightarrow 0^+} \frac{\Phi(tx(s_1), \dots, tx(s_n))}{t} = \Phi_0(\delta_{s_1}, \dots, \delta_{s_n})(x).$$

So the limit exists and, since Φ_0 is a lattice-linear expression, $Pf \in \text{FVL}(S)$. From the formula defining Pf it is immediate to check that the map $f \mapsto Pf$ is a lattice homomorphic projection extending δ . By uniqueness, it must be the projection introduced above.

For lattice-linear expressions, there is continuity with respect to the expressions in the following sense: if $(\Phi_k[t_1, \dots, t_n])_k$ is a sequence of lattice-linear expressions that converges uniformly to $\Phi[t_1, \dots, t_n]$ when seen as functions in $C([-1, 1]^n)$, then $\Phi_k(x_1, \dots, x_n)$ converges in norm to $\Phi(x_1, \dots, x_n)$ for every $x_1, \dots, x_n$ in a Banach lattice X . This is no longer true for LLA expressions.

Example 8.16. Consider the sequence of LLA expressions

$$\Phi_k[t] = t \left(\sum_{j=0}^k \frac{t^{2j}}{(2j)!} \right)^2 - t \left(\sum_{j=0}^k \frac{t^{2j+1}}{(2j+1)!} \right)^2.$$

For every $x \in [-1, 1]$:

$$\lim_{k \rightarrow \infty} \Phi_k(x) = x \cosh^2 x - x \sinh^2 x = x$$

and this limit is uniform on $[-1, 1]$. In other words, if $\Phi[t] = t$, and we see Φ_k and Φ as elements in $C([-1, 1])$, $\Phi_k \rightarrow \Phi$ in the uniform norm. But, in $\mathbb{R}_0$ (or any other Banach lattice with the zero product), $\Phi_k(x) = 0$ whereas $\Phi(x) = x$ for every $x \in \mathbb{R}_0$. Hence $\Phi_k(x)$ does not converge to $\Phi(x)$ if $x \neq 0$.

8.1.3 Algebraic properties

In this section the algebraic properties of $\text{FAfA}(S)$ are considered. Recall that an f -algebra A is *semiprime* if $N(A) = \{0\}$.

Proposition 8.17. *Let S be a set.*

- (i) $\text{FAfA}(S)$ is semiprime.
- (ii) $\text{FAfA}(S)$ does not have an identity element.

Proof. The result is trivial when S is empty. For the rest of the proof, assume that S is non-empty.

- (i) Suppose $f \in N(\text{FAfA}(S))$, and let $x \in \mathbb{R}^S$ be different from zero. There exists $s \in S$ such that $x(s) \neq 0$. Then $0 = (f\delta_s)(x) = f(x)x(s)$, so $f(x) = 0$. This shows that $f = 0$.
- (ii) Suppose $1 \in \text{FAfA}(S)$ were an identity. Fix $s_0 \in S$ and define $T: S \rightarrow C[-1, 1]$ by $Ts = 0$ for $s \neq s_0$ and $(Ts_0)(t) = t_+$ for $t \in [-1, 1]$. Then $\hat{T}(1) = \hat{T}(1)^2$. Since $[-1, 1]$ is connected, $\hat{T}(1)$ is either 0 or $\mathbb{1}_{[-1, 1]}$, the constant one function. But $\hat{T}(1)\hat{T}(\delta_{s_0}) = \hat{T}(\delta_{s_0}) = Ts_0 \neq 0$, so $\hat{T}(1) = \mathbb{1}_{[-1, 1]}$. This means that $\mathbb{1}_{[-1, 1]} \in \hat{T}(\text{FAfA}(S)) = \text{VLA}\{Ts_0\}$. This is impossible because every element of $\text{VLA}\{Ts_0\}$ has support contained in $[0, 1]$. $\square$

It is natural to wonder whether a chain condition for the product exists. Namely, if $F \subseteq \text{FAfA}(S)$ is an uncountable family, is it true that there exist

$f, g \in F$ such that $fg \neq 0$? To answer this question, it is useful to reformulate semiprimeness in terms of what Scheffold [Sch81] calls f^* -algebras: a lattice-ordered algebra A is called an f^* -algebra when

$$ab = 0 \text{ if and only if } |a| \wedge |b| = 0 \text{ for all } a, b \in A.$$

The following is part of [BKW77, Théorème 9.3.1].

Lemma 8.18. *An f -algebra A is an f^* -algebra if and only if it is semiprime.*

Remark 8.19. In particular, $\text{FA}fA(S)$ is an f^* -algebra. Therefore, families $F \subseteq \text{FA}fA(S)$ such that $ab = 0$ for all distinct $a, b \in F$ are the same as families of pairwise disjoint elements, and therefore must be countable.

8.2 A STRUCTURE THEOREM FOR BANACH f -ALGEBRAS

8.2.1 A structure theorem

Before getting to free Banach f -algebras, the main topic of this chapter, we need to introduce a new tool in the theory of Banach f -algebras. Recall that every vector lattice X , when equipped with the identically 0 product, becomes an f -algebra. We are denoting this f -algebra by X_0 . In particular, when X is a Banach lattice, X_0 is a Banach f -algebra. Another basic example of Banach f -algebra is the space $C(K)$ of continuous functions on a compact Hausdorff space K , with the linear operations, order and product defined pointwise. Given Banach f -algebras A and B their direct sum $A \oplus_\infty B$, equipped with the maximum norm and coordinatewise operations, is also a Banach f -algebra.

The next theorem shows that every Banach f -algebra can be seen as a (not necessarily closed) sublattice-algebra of $X_0 \oplus_\infty C(K)$, for appropriate X and K .

Theorem 8.20. *For every Banach f -algebra A there exist a Banach lattice X , a compact Hausdorff space K and a contractive injective lattice-algebra homomorphism $R: A \rightarrow X_0 \oplus_\infty C(K)$.*

The approach to the proof will be very similar to that of Theorem 8.2. First we consider the case that A has trivial annihilator. The following was already proved in [Mar80, Proposition 1.7]. Since the proof uses some ideas that are worth keeping in mind, we sketch it below.

Lemma 8.21. *For every Banach f -algebra A with trivial annihilator there exists a contractive injective lattice-algebra homomorphism $A \rightarrow C(K)$, for a certain compact Hausdorff space K .*

Proof. Since A is a Banach lattice, $\text{Orth}(A)$ coincides with the center $\mathcal{Z}(A)$ of A . The map $L: A \rightarrow \mathcal{Z}(A)$ sending $a \in A$ to L_a , the operator of left multiplication by a , is an injective lattice-algebra homomorphism. And the center $\mathcal{Z}(A)$, being a Banach f -algebra with strong unit id_A , is lattice-algebra isometric to $C(K)$ for some compact Hausdorff space K (Theorem 1.8). $\square$

The map in the previous lemma need not be an embedding, as the following example shows.

Example 8.22. Consider ℓ_1 with the usual Banach lattice structure and point-wise product. This is a Banach f -algebra. Note that, given $x \in \ell_1$, the norm of L_x is

$$\|L_x\| = \sup_{y \in B_{\ell_1}} \|xy\|_1 = \|x\|_\infty.$$

Hence the map $L: \ell_1 \rightarrow \mathcal{Z}(\ell_1)$ is not an embedding.

In view of the previous example, one could be led to think that the embedding fails because we are taking a norm that is “much bigger” than a supremum norm. The following example shows that even for norms defined as a supremum the map L can fail to be an embedding.

Example 8.23. Let $(a_n) \subseteq \mathbb{R}$ be a sequence such that $a_n \geq 1$ and $a_n \rightarrow \infty$. Consider the space of sequences

$$A = \{ (x_n) \in \mathbb{R}^{\mathbb{N}} : x_n a_n \rightarrow 0 \}.$$

This is a sublattice of $\mathbb{R}^{\mathbb{N}}$ that is closed under the coordinatewise product. Equipped with the norm

$$\|(x_n)\| = \sup_n |x_n| a_n$$

it becomes a Banach lattice. Since

$$\|xy\| = \sup_n |x_n| |y_n| a_n \leq \sup_n |x_n| |y_n| a_n^2 \leq \|x\| \|y\|,$$

A is a Banach f -algebra. The sequence (e_n/a_n) is such that $\|e_n/a_n\| = 1$. If $x \in A$ is in the unit ball (that is, $|x_n| \leq 1/a_n$ for all $n \in \mathbb{N}$), then

$$\|e_n/a_n x\| = \frac{1}{a_n} \|x_n e_n\| = |x_n| \leq \frac{1}{a_n}$$

which tends to 0 as $n \rightarrow \infty$. Therefore $\|L_{e_n/a_n}\|$ tends to zero while $\|e_n/a_n\| = 1$.

Proof of Theorem 8.20. By embedding A in its bidual, which is again a Banach f -algebra when equipped with the Arens product, we may assume A is order complete. Let $P: A \rightarrow A$ denote the band projection onto its annihilator $N = N(A)$, and let P^d denote the band projection onto its disjoint complement. Both P and P^d are multiplicative (see the proof of Theorem 8.2). In particular, both N and N^d are Banach f -algebras on their own, with the structure inherited from A .

The map

$$\begin{aligned} T: \quad A &\longrightarrow N \oplus_\infty N^d \\ f &\longmapsto (Pf, P^d f) \end{aligned}$$

defines an injective lattice-algebra homomorphism. Since both P and P^d are contractive, so is T . Note that N is a Banach lattice with the zero product. On the other hand, N^d is a Banach f -algebra with trivial annihilator. By

Lemma 8.21 there exists an injective and contractive lattice-algebra homomorphism $S: N^d \rightarrow C(K)$. The composition

$$R = (\text{id}_N \oplus S)T: A \rightarrow N \oplus_\infty C(K)$$

defines the desired injective and contractive lattice-algebra homomorphism. $\square$

Remark 8.24. (i) In Theorem 8.20, R need not be an embedding, as Example 8.22 shows.

- (ii) Passing to the bidual in the proof of Theorem 8.20 is not superfluous, for the annihilator need not be a projection band when the Banach f -algebra is not order complete. For instance, consider in the Banach lattice $C[0, 1]$ the element

$$p(t) = (2t - 1)\chi_{[1/2, 1]}, \quad t \in [0, 1],$$

and define, for $f, g \in C[0, 1]$, the product $f \star g = fgp$. It is clear that $C[0, 1]$ with this product becomes a Banach f -algebra, with annihilator

$$N = \{ f \in C[0, 1] : \text{supp}(f) \subseteq [0, 1/2] \}.$$

This is certainly a band but it is not a projection band.

- (iii) Theorem 8.20 is not true for general f -algebras, even if they are Archimedean and uniformly complete. For instance, $\mathbb{R}^\mathbb{N}$ with the pointwise order and product is an Archimedean and uniformly complete f -algebra with trivial annihilator, yet there is no positive map from $\mathbb{R}^\mathbb{N}$ to a space of continuous functions. Indeed, if $R: \mathbb{R}^\mathbb{N} \rightarrow C(K)$ were positive, where K is some compact Hausdorff space, then the sequence $x = (n/\|Re_n\|_\infty)_n$ would satisfy

$$Rx \geq \frac{n}{\|Re_n\|_\infty} Re_n$$

so that $\|Rx\|_\infty \geq n$ for every $n \in \mathbb{N}$. This is a contradiction.

- (iv) However, Theorem 8.20 does hold for normed f -algebras: just embed them in their norm completion, which is a Banach f -algebra (see [Jab20, Proposition 2.1]).
- (v) Given a Banach f -algebra A and a map $R: A \rightarrow X_0 \oplus_\infty C(K)$ as in Theorem 8.20, we shall denote $Ra = (a_N, a_C)$ for $a \in A$. Note that the maps $a \mapsto a_N$ and $a \mapsto a_C$ are also contractive lattice-algebra homomorphisms.

The following theorem is a refinement of Theorem 8.2 for normed f -algebras.

Theorem 8.25. *Let Φ be an LLA expression. If Φ vanishes on $[-1, 1]$, then it also vanishes on the unit ball of every normed f -algebra.*

Proof. Since Φ vanishes on a neighbourhood of 0, it must vanish on $\mathbb{R}_0$ (Lemma 8.5). Since Φ vanishes on $\mathbb{R}_0$, it must also vanish on every vector lattice with the zero product (Lemma 8.4). Let A be a Banach f -algebra, and

let $a_1, \dots, a_n \in A$ be such that $\|a_i\| \leq 1$. Let $R: A \rightarrow X_0 \oplus_\infty C(K)$ be a map as in Theorem 8.20. Then

$$\Phi(a_1, \dots, a_n)_N = \Phi((a_1)_N, \dots, (a_n)_N) = 0.$$

Since R is contractive, $\|(a_i)_C\|_\infty \leq \|Ra_i\| \leq \|a_i\| \leq 1$, so $|(a_i)_C(t)| \leq 1$ for every $t \in K$. By assumption:

$$\Phi((a_1)_C(t), \dots, (a_n)_C(t)) = 0 \quad \text{for every } t \in K.$$

This yields $0 = \Phi((a_1)_C, \dots, (a_n)_C) = \Phi(a_1, \dots, a_n)_C$. Thus $R(\Phi(a_1, \dots, a_n)) = (0, 0)$. Injectivity of R implies $\Phi(a_1, \dots, a_n) = 0$. $\square$

Remark 8.26. The previous result is meaningless for LL expressions: if an LL expression vanishes on $[-1, 1]$, then by positive homogeneity it vanishes on $\mathbb{R}$. This is no longer true for LLA expressions since, in general, they are not positively homogeneous. For instance,

$$\Phi[t] = (t_+^2 - t_+)_+$$

is an LLA expression that vanishes on $[-1, 1]$ but not on $\mathbb{R}$. By the previous theorem, the identity $(x_+^2 - x_+)_+ = 0$ holds whenever x is contained in the unit ball of a normed f -algebra.

8.2.2 Minimal representation of semiprime f -algebras

In the case that a Banach f -algebra A is semiprime, Theorem 8.20 yields a contractive lattice-algebra homomorphism $A \rightarrow C(K)$, for some compact Hausdorff space K . We are going to show that there exists a unique K that is minimal in a certain sense, and such that the map $A \rightarrow C(K)$ has good extension properties. This minimal compact Hausdorff space will be the set of functionals on A that are lattice-algebra homomorphisms.

Definition 8.27. Let A be a Banach f -algebra. We will denote by $K_A \subseteq A^*$ the set of functionals on A that are lattice-algebra homomorphisms.

Lemma 8.28. *Let A be a semiprime Banach f -algebra. The set K_A , equipped with the weak* topology, is a compact Hausdorff space that separates the points of A .*

Proof. If $\phi \in K_A$ and $a \in A$, then

$$|\phi(a)|^2 = \phi(a^2) \leq \|a\|\phi(|a|) = \|a\||\phi(a)|,$$

where we have used that $a^2 \leq \|a\||a|$ (recall that this holds because left multiplication by $|a|$ is a central operator with norm no bigger than $\|a\|$). Hence $\|\phi\| \leq 1$, and $K_A \subseteq B_{A^*}$. It is then immediate to check that K_A is closed in the weak* topology.

To show that K_A separates the points of A , let $a, b \in A$ be different points. By the arguments of Theorem 8.20, there exists a contractive and injective lattice-algebra homomorphism $R: A \rightarrow C(K)$, for some compact Hausdorff space K . Since $Ra \neq Rb$, there exists a point $t \in K$ at which $(Ra)(t) \neq (Rb)(t)$. Then $R^*\delta_t \in K_A$, where δ_t denotes the point evaluation at t , is such that $(R^*\delta_t)(a) \neq (R^*\delta_t)(b)$. $\square$

Proposition 8.29. *Let A be a semiprime Banach f -algebra. Then the map $R: A \rightarrow C(K_A)$, that sends $a \in A$ to its evaluation functional, is a contractive and injective lattice-algebra homomorphism. Moreover, both the vector lattice and the algebra generated by $\{Ra : a \in A\} \cup \{1_{K_A}\}$ in $C(K_A)$ are dense.*

Proof. Injectivity follows from Lemma 8.28. Density of both the vector lattice and the algebra generated by $\{Ra : a \in A\} \cup \{1_{K_A}\}$ is a direct consequence of the Stone–Weierstrass theorem. $\square$

The following example illustrates this construction and shows that adding the constant one function is necessary in order to obtain density.

Example 8.30. Consider c_0 with the usual order and product. The lattice-algebra homomorphisms on c_0 are the δ_i , the projection onto the i coordinate, and the zero functional. It is immediate to check that K_{c_0} is homeomorphic to $\mathbb{N} \cup \{\infty\}$, and therefore that $C(K_{c_0})$ can be identified with c , the space of convergent sequences. The canonical map $R: c_0 \rightarrow c$ is the usual inclusion.

8.2.3 Extension of lattice-algebra homomorphisms and injective objects

In this section we show the remarkable fact that, if A is a closed sublattice-algebra of a Banach f -algebra B , then every lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ extends to a lattice-algebra homomorphism on B . Notice that, for such a real-valued lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$, $\phi(x) = 0$ for every $x \in N(A)$. Hence $\phi|_{N(A)} = 0$ and, to truly understand ϕ and its extensions, we need only understand its restriction to the semiprime Banach f -algebra $N(A)^d$. For this reason, our focus is in the case that A is semiprime.

So let A be a semiprime Banach f -algebra, and let $R: A \rightarrow C(K_A)$ be the map that sends each element of A to its evaluation functional. Since the non-zero lattice-algebra homomorphisms on $C(K_A)$ are precisely the point evaluations, it follows from the construction that each non-zero lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ extends uniquely to a lattice-algebra homomorphism $\tilde{\phi}: C(K_A) \rightarrow \mathbb{R}$, in the sense that $\phi = \tilde{\phi}R$. The zero homomorphism extends in precisely two ways: as the zero homomorphism, and as the point evaluation at $0 \in K_A$. The following shows that this extension property characterizes K_A .

Proposition 8.31. *Let A be a semiprime Banach f -algebra. Let $R: A \rightarrow C(K)$ be a contractive and injective lattice-algebra homomorphism, for some compact Hausdorff space K . Suppose that every lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ extends uniquely to a non-zero lattice-algebra homomorphism $\tilde{\phi}: C(K) \rightarrow \mathbb{R}$, in the sense that $\phi = \tilde{\phi}R$. Then K is homeomorphic to K_A .*

Proof. Consider the map $K \rightarrow K_A$ that sends each $t \in K$ to $R^*\delta_t \in K_A$. This map is certainly continuous. It is also bijective, because for every $\phi \in K_A$, the unique non-zero extension $\tilde{\phi}: C(K) \rightarrow \mathbb{R}$ is of the form $\tilde{\phi} = \delta_t$ for some $t \in K$, and t is precisely the unique preimage of ϕ through this map. Since K and K_A are compact, this map is a homeomorphism. $\square$

Proposition 8.32. *Let A be a semiprime Banach f -algebra. Let $R: A \rightarrow C(K)$ be a contractive and injective lattice-algebra homomorphism, for some compact Hausdorff space K . Suppose that every lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ extends to a non-zero lattice-algebra homomorphism $\tilde{\phi}: C(K) \rightarrow \mathbb{R}$, in the sense that $\phi = \tilde{\phi}R$. Suppose also that the sublattice-algebra generated by $\{Ra : a \in A\} \cup \{\mathbb{1}_K\}$ is dense in $C(K)$. Then K is homeomorphic to K_A .*

Proof. By Proposition 8.31, we need only show that the (non-zero) extension of every lattice-algebra homomorphism ϕ is unique. Suppose ψ and ψ' are two non-zero lattice-algebra homomorphisms that satisfy $\psi R = \psi' R$. Since every non-zero lattice-algebra homomorphism on $C(K)$ is a point evaluation, $\psi(\mathbb{1}_{K_A}) = 1 = \psi'(\mathbb{1}_{K_A})$. Thus ψ and ψ' coincide on the sublattice-algebra generated by $\{Ra : a \in A\} \cup \{\mathbb{1}_{K_A}\}$. By density, $\psi = \psi'$. $\square$

It is remarkable that the center of a Banach f -algebra provides an instance of a $C(K)$ space to which every functional that is a lattice-algebra homomorphism extends.

Proposition 8.33. *Let A be a Banach f -algebra, and let $L: A \rightarrow \mathcal{Z}(A)$ be the map that sends every element of A to its left multiplication operator. For every lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ there exists a lattice-algebra homomorphism $\tilde{\phi}: \mathcal{Z}(A) \rightarrow \mathbb{R}$ satisfying $\phi = \tilde{\phi}L$. Moreover, if A is semiprime and the closed sublattice-algebra generated by $\{La : a \in A\}$ in $\mathcal{Z}(A)$ does not contain id_A , then $C(K_A)$ can be identified with the closed sublattice-algebra with identity of $\mathcal{Z}(A)$ generated by $\{La : a \in A\}$.*

Proof. The zero homomorphism clearly satisfies the statement, so assume that $\phi \neq 0$. Then there exists $x_0 \in A_+$ such that $\phi(x_0) = 1$. Define the map

$$\begin{array}{ccc} \tilde{\phi}: & \mathcal{Z}(A) & \longrightarrow \mathbb{R} \\ & T & \longmapsto \phi(Tx_0) \end{array} .$$

First note that, for every $a \in A$,

$$\tilde{\phi}(L_a) = \phi(L_a(x_0)) = \phi(ax_0) = \phi(a)\phi(x_0) = \phi(a),$$

so $\phi = \tilde{\phi}L$. It is immediate that $\tilde{\phi}$ is linear. It is also a lattice homomorphism, because

$$\tilde{\phi}(|T|) = \phi(|T|x_0) = \phi(|Tx_0|) = |\phi(Tx_0)| = |\tilde{\phi}(T)|,$$

where in the second equality we are using that the modulus of a central operator is computed pointwise, and in the third that ϕ is a lattice homomorphism. We also have $\tilde{\phi}(\text{id}_A) = 1$. Since $\mathcal{Z}(A)$ is lattice-algebra isometric to a space of

continuous functions on a compact Hausdorff space, every functional that is a lattice homomorphism and sends the identity to 1 is a point evaluation, and therefore an algebra homomorphism. Hence $\tilde{\phi}$ is the desired extension.

Suppose that $\mathcal{A}$, the closed sublattice-algebra generated by $\{La : a \in A\}$ in $\mathcal{Z}(A)$, does not contain id_A . Then there exists a point evaluation on $\mathcal{Z}(A)$ that vanishes on $\mathcal{A}$. This point evaluation is a non-zero extension of the zero functional on $\mathcal{A}$. Hence, by Proposition 8.32, the closed sublattice-algebra generated by $\{La : a \in A\} \cup \{\text{id}_A\}$ in $\mathcal{Z}(A)$, which is again a space of continuous functions, can be identified with $C(K_A)$. $\square$

These properties can now be used to prove the main extension result.

Theorem 8.34. *Let B be a Banach f -algebra, and let $A \subseteq B$ be a closed sublattice-algebra. Then for every lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$ there exists a lattice-algebra homomorphism $\psi: B \rightarrow \mathbb{R}$ such that $\psi|_A = \phi$.*

Proof. As usual, denote by $L_b: B \rightarrow B$ the multiplication operator defined by $b \in B$. Note that, for every $a \in A$, the operator L_a leaves A invariant and its restriction to A is precisely the multiplication operator defined by a in A . Let $\mathcal{A}$ be the sublattice-algebra with identity generated by $\{L_a|_A : a \in A\}$ inside $\mathcal{Z}(A)$.

Let $\mathcal{A}'$ be the sublattice-algebra with identity generated by $\{L_a : a \in A\}$ inside $\mathcal{Z}(B)$. We claim that every element of $\mathcal{A}'$ leaves A invariant, and that the elements of $\mathcal{A}$ are precisely the restriction of elements of $\mathcal{A}'$ to A . Indeed, if $T, S \in \mathcal{Z}(B)$ leave A invariant, then:

- (i) $\alpha T + \beta S$ leaves A invariant, for every $\alpha, \beta \in \mathbb{R}$, and $(\alpha T + \beta S)|_A = \alpha T|_A + \beta S|_A$.
- (ii) TS leaves A invariant, and $(TS)|_A = T|_A S|_A$.
- (iii) $|T|$ leaves A invariant, and $(|T|)|_A = |(T|_A)|$. Indeed, if $x \in A_+ \subseteq B_+$, then since the lattice operations on central operators are computed point-wise both on A_+ and B_+ :

$$|T|_A(x) = |T|(x) = |Tx| = |T|_A x = |T|_A|(x).$$

In other words, we have shown that the map

$$\begin{aligned} \mathcal{R}: \mathcal{A}' &\longrightarrow \mathcal{A} \\ T &\longmapsto T|_A, \end{aligned}$$

is a lattice-algebra homomorphism. Moreover, if we equip $\mathcal{A}$ with the norm of $\mathcal{Z}(A)$ (i.e., with the uniform norm $\|\cdot\|_{\text{id}_A}$ induced by the identity id_A on A), and $\mathcal{A}'$ with the norm of $\mathcal{Z}(B)$ (i.e., with the uniform norm $\|\cdot\|_{\text{id}_B}$ induced by the identity id_B on B), then $\mathcal{R}$ is contractive. Indeed, if $|T| \leq \|T\|_{\text{id}_B} \text{id}_B$, then taking restrictions to A , we have

$$|T|_A = |T||_A \leq \|T\|_{\text{id}_B} \text{id}_B|_A = \|T\|_{\text{id}_B} \text{id}_A.$$

Hence $\|T|_A\|_{\text{id}_A} \leq \|T\|_{\text{id}_B}$, and $\mathcal{R}$ extends naturally to the closures $\mathcal{R}: \overline{\mathcal{A}'} \rightarrow \overline{\mathcal{A}}$.

Let $\phi: A \rightarrow \mathbb{R}$ be a lattice-algebra homomorphism. From Proposition 8.33 it follows that there exists a lattice-algebra homomorphism $\psi: \overline{\mathcal{A}} \rightarrow \mathbb{R}$ such that $\phi = \psi L$. Then $\psi \circ \mathcal{R}: \overline{\mathcal{A}'} \rightarrow \mathbb{R}$ is a lattice-algebra homomorphism satisfying

$$(\psi \circ \mathcal{R})(L_a) = \psi(L_a|_A) = \phi(a).$$

Since $\overline{\mathcal{A}'}$ is a closed subalgebra with identity of $\mathcal{Z}(B)$, which can be identified with a space of continuous functions, $\psi \circ \mathcal{R}$ further extends to a lattice-algebra homomorphism $\tilde{\psi}$ on $\mathcal{Z}(B)$. The composition $\tilde{\phi} = \tilde{\psi} L: B \rightarrow \mathbb{R}$ is the claimed extension. $\square$

The extensions in Theorem 8.34 do not, in general, preserve the norm.

Example 8.35. Consider ℓ_1^2 with pointwise order and product. This is a Banach f -algebra. Let $A = \text{span}\{(1, 1)\}$. It is clear that A is a sublattice-algebra of ℓ_1^2 . The map $\phi: A \rightarrow \mathbb{R}$ defined by $\phi(\lambda, \lambda) = \lambda$, $\lambda \in \mathbb{R}$, is a lattice-algebra homomorphism. If $(\lambda, \lambda) \in A$, and $\|(\lambda, \lambda)\|_1 \leq 1$, then $|\lambda| \leq 1/2$ and therefore $\|\phi\| \leq 1/2$.

The non-zero lattice-algebra homomorphisms on ℓ_1^2 are the projections onto the coordinates. Both of these functionals are extensions of ϕ , yet they have norm one.

It is pertinent now to introduce, in the category of Banach f -algebras, the analogue of the notion of λ -injective Banach lattice mentioned in Section 1.6.3.

Definition 8.36. Let $\lambda \geq 1$. A Banach f -algebra I is λ -injective if, for every Banach f -algebra A , every closed sublattice-algebra B of A , and every lattice-algebra homomorphism $T: B \rightarrow I$, there exists an extension $\tilde{T}: A \rightarrow I$ that is also a lattice-algebra homomorphism and satisfies $\|\tilde{T}\| \leq \lambda \|T\|$.

Example 8.35 shows that $\mathbb{R}$ is not 1-injective in the category of Banach f -algebras. This example can be generalized to show the following.

Theorem 8.37. For $\lambda \geq 1$, there are no non-zero λ -injective Banach f -algebras.

For the proof of this theorem we will need the following elementary lemma.

Lemma 8.38. Let Γ be a set and let $\{X_\gamma\}_{\gamma \in \Gamma}$ be a family of Banach lattices. Let X be their ℓ_1 -sum, and let $p_\gamma: X \rightarrow X_\gamma$ denote the projection onto the $\gamma \in \Gamma$ coordinate. Let $\phi: X \rightarrow \mathbb{R}$ be a lattice homomorphism. Then there exists a $\delta \in \Gamma$ and a lattice homomorphism $\phi': X_\delta \rightarrow \mathbb{R}$ such that $\phi = \phi' p_\delta$.

Proof. The result is obvious for $\phi = 0$, so assume $\phi \neq 0$. Let $(x_\gamma) \in X_+$ be such that $\phi((x_\gamma)) > 0$. For $x \in X_\gamma$, denote by $x e_\gamma$ the element of X that takes the value x at $\gamma \in \Gamma$ and zero everywhere else. Since $(x_\gamma) = \sum_{\gamma \in \Gamma} x_\gamma e_\gamma$ and $0 \leq x_\gamma e_\gamma$ for every $\gamma \in \Gamma$, there must exist some $\delta \in \Gamma$ for which $\phi(x_\delta e_\delta) > 0$. Now if $y_\gamma \in (X_\gamma)_+$ with $\gamma \neq \delta$, then $(x_\delta e_\delta) \wedge (y_\gamma e_\gamma) = 0$, and since ϕ is a lattice

homomorphism, $\phi(x_\delta e_\delta) \wedge \phi(y_\gamma e_\gamma) = 0$. Thus $\phi(y_\gamma e_\gamma) = 0$. Now for a general $(y_\gamma) \in X$:

$$\phi(y_\gamma) = \phi\left(\sum_{\gamma \in \Gamma} y_\gamma e_\gamma\right) = \sum_{\gamma \in \Gamma} \phi(y_\gamma e_\gamma) = \phi(y_\delta e_\delta).$$

If we define $\phi': X_\delta \rightarrow \mathbb{R}$ by $\phi'(x) = \phi(xe_\delta)$, then it is immediate to check that ϕ' is a lattice homomorphism satisfying $\phi = \phi'p_\delta$. $\square$

Lemma 8.39. *Let A be a Banach f -algebra. Suppose that the product on A is not identically zero. Then there exists a non-zero lattice-algebra homomorphism $\phi: A \rightarrow \mathbb{R}$.*

Proof. By Theorem 8.20 there exist a non-empty compact Hausdorff space K , a Banach lattice X , and a contractive injective lattice-algebra homomorphism $R: A \rightarrow X_0 \oplus_\infty C(K)$. Projecting onto the second coordinate, we obtain a lattice-algebra homomorphism $R_2: A \rightarrow C(K)$. Since K is non-empty (because the product on A is not identically zero), there exists a non-zero lattice-algebra homomorphism $\delta_t: C(K) \rightarrow \mathbb{R}$ (a point evaluation at some $t \in K$). Hence the composition $\delta_t R_2: A \rightarrow \mathbb{R}$ is a non-zero lattice-algebra homomorphism. $\square$

Proof of Theorem 8.37. Suppose that I is a non-zero λ -injective Banach f -algebra, for some $\lambda \geq 1$. First note that the product in I cannot be identically zero, for if it were, then I would be λ -injective for lattice homomorphisms (if X is a Banach lattice, and $T: X \rightarrow I$ is a lattice homomorphism, then $T: X_0 \rightarrow I_0$ is a lattice-algebra homomorphism). But, as explained in Section 1.6.3, there are no non-zero λ -injective objects in the category whose objects are Banach lattices and whose morphisms are lattice homomorphisms.

Choose $n \in \mathbb{N}$ with $n > \lambda$. Let B be the ℓ_1 -sum of n copies of I , and let

$$A = \{(x, \dots, x) : x \in I\}.$$

It is immediate to check that A is a closed sublattice-algebra of B . Let $\phi: A \rightarrow I$ be defined by $\phi(x, \dots, x) = x$ for every $x \in I$. This is a lattice-algebra homomorphism with $\|\phi\| = 1/n$; indeed, if $\|(x, \dots, x)\|_1 = 1$, then $\|x\| = 1/n$.

Since I is λ -injective, this map extends to a lattice-algebra homomorphism $\tilde{\phi}: B \rightarrow I$ with $\|\tilde{\phi}\| \leq \lambda/n$. Since the product in I is not identically zero, by Lemma 8.39 there exists a non-zero lattice-algebra homomorphism $\psi: I \rightarrow \mathbb{R}$. Then the composition $\psi\tilde{\phi}: B \rightarrow \mathbb{R}$ is a lattice homomorphism which, by Lemma 8.38, is of the form $\psi\tilde{\phi} = \phi'p_i$ for some lattice homomorphism $\phi': I \rightarrow \mathbb{R}$, where $p_i: B \rightarrow I$ denotes the projection onto the i coordinate. Now note that, for every $x \in I$,

$$\phi'(x) = (\psi\tilde{\phi})(x, \dots, x) = (\psi\phi)(x, \dots, x) = \psi(x).$$

Thus $\psi\tilde{\phi} = \psi p_i$. It is now immediate to check that $\|\psi p_i\| = \|\psi\|$, so that

$$\|\psi\| = \|\psi\tilde{\phi}\| \leq \|\psi\|\|\tilde{\phi}\| \leq \|\psi\|\frac{\lambda}{n}.$$

Since $\|\psi\| \neq 0$, $n \leq \lambda$, which contradicts the initial choice of $n \in \mathbb{N}$. $\square$

Remark 8.40. According to Theorem 8.34, in Banach f -algebras we can always extend real-valued lattice-algebra homomorphisms but, according to Theorem 8.37, we can never do so with a uniform control over the norm.

8.3 THE FREE BANACH f -ALGEBRA GENERATED BY A BANACH SPACE

The notion of a free Banach lattice algebra has been considered before [Wic17b, dJ21]. Through abstract arguments from universal algebras, it is possible to show that such an object exists. Yet no explicit construction is known, and it is not clear at all whether such an explicit description is even possible.

The goal of this section is to study free objects in the much better behaved category of Banach f -algebras and to get as close as we can to an explicit description of these spaces. We shall see that, even though a large theory of free Banach lattices is known (see Section 1.6 and the references therein), free Banach f -algebras are still challenging to understand. New ideas are required, since the introduction of a product invalidates arguments that rely on positive homogeneity. In fact, a first subtlety due to the lack of positive homogeneity is already present in the definition of free Banach f -algebra.

Definition 8.41. Let E be a Banach space. The *free Banach f -algebra generated by E* is a Banach f -algebra $\text{FBfA}[E]$ together with a linear isometric embedding $\eta_E: E \rightarrow \text{FBfA}[E]$ such that, for every Banach f -algebra A and every contractive operator $T: E \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FBfA}[E] \rightarrow A$ satisfying $\hat{T} \circ \eta_E = T$ and $\|\hat{T}\| = \|T\|$.

Remark 8.42. The universal property defining $\text{FBfA}[E]$ only extends contractive operators. The reason for this is that non-contractive maps cannot, in general, be extended to bounded algebra homomorphisms. For example, if the map $T: \mathbb{R} \rightarrow \mathbb{R}$, $Tx = 2x$, extended to a lattice-algebra homomorphism $\hat{T}: \text{FBfA}[\mathbb{R}] \rightarrow \mathbb{R}$, then

$$\hat{T}(\eta_{\mathbb{R}}(1)^n) = T(1)^n = 2^n$$

while $\|\eta_{\mathbb{R}}(1)^n\| \leq \|\eta_{\mathbb{R}}(1)\|^n \leq 1$. This would imply that $\hat{T}$ is unbounded. Another way to see this obstruction is that, while scaling a lattice homomorphism by a positive number gives another lattice homomorphism, scaling an algebra homomorphism does not give an algebra homomorphism (unless the scalar is 1).

Remark 8.43. Let S be a set. We shall denote $\text{FBfA}[\ell_1(S)]$ by $\text{FBfA}(S)$ and call it the *free Banach f -algebra generated by S* . The reason for this is that for every Banach f -algebra A and every map $T: S \rightarrow A$ satisfying $\|T\| = \sup_{s \in S} \|Ts\| \leq 1$ there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FBfA}(S) \rightarrow A$ with $\|\hat{T}\| = \|T\|$ satisfying $\hat{T}\eta_{e_s} = Ts$.

The existence of the free Banach f -algebra generated by a Banach space follows from that of $\text{FAfA}(S)$ using abstract general arguments that are already known. The details of this construction are carried out in Section 8.3.1. But, as we have anticipated, these arguments do not provide an explicit description of

the space. In Section 8.3.2 we give a more explicit and useful description of free Banach f -algebras using Theorem 8.25. In Section 8.3.3 a characterization, up to isomorphism, of the free Banach f -algebra generated by a finite-dimensional Banach space is derived. Finally, Section 8.3.4 is devoted to the study of the free norm; it will be particularly useful when we study the properties of free Banach f -algebras.

Although the following objects are not central, it is convenient to have them at hand, since they appear as intermediate steps in the construction.

Definition 8.44. Let E be a vector space. The *free Archimedean f -algebra generated by E* is an Archimedean f -algebra $\text{FA}fA[E]$ together with a linear map $\delta_E: E \rightarrow \text{FA}fA[E]$ such that, for every Archimedean f -algebra A and every linear map $T: E \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FA}fA[E] \rightarrow A$ satisfying $\hat{T} \circ \delta_E = T$.

Definition 8.45. Let E be a normed space. The *free normed f -algebra generated by E* is a normed f -algebra $\text{FN}fA[E]$ together with a linear isometric embedding $\eta_E: E \rightarrow \text{FN}fA[E]$ such that, for every normed f -algebra A and every contractive operator $T: E \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FN}fA[E] \rightarrow A$ satisfying $\hat{T} \circ \eta_E = T$ and $\|\hat{T}\| = \|T\|$.

The reason to have that much notation is the following: through the universal properties, it is not difficult to check that $\text{FA}fA[E]$ is nothing more than $\text{FA}fA(S)$, where S is a Hamel basis of E . We will equip $\text{FA}fA[E]$ with an appropriate seminorm, and quotient out by its kernel to obtain a norm. This will yield $\text{FN}fA[E]$. It is easy to check, again using universal properties, that the completion of $\text{FN}fA[E]$ is precisely $\text{FB}fA[E]$. With these notations we are aware, at each step of the process, which universal property each object has. Furthermore, we will also exhibit an explicit description of these free objects.

8.3.1 Abstract construction

We shall follow the previous outline to show that $\text{FB}fA[E]$ exists. In doing so, we follow closely the expositions of V. Troitsky [Tro19] and M. de Jeu [dJ21]. The construction of $\text{FA}fA[E]$ that we use is essentially the same, with the appropriate adaptations, as that of [Tro19] for the free vector lattice over a vector space.

Let E be a vector space, and let $E^\#$ be its algebraic dual. The space of functions $\mathbb{R}^{E^\#}$, equipped with pointwise linear operations, order and product, is an Archimedean f -algebra. For every $x \in E$, define $\delta_x: E^\# \rightarrow \mathbb{R}$ by $\delta_x(\omega) = \omega(x)$ for all $\omega \in E^\#$. Denote $\text{FA}fA[E] = \text{VLA}\{\delta_x : x \in E\} \subseteq \mathbb{R}^{E^\#}$, and define the map $\delta: E \rightarrow \text{FA}fA[E]$ by $\delta(x) = \delta_x$.

Proposition 8.46. *The free Archimedean f -algebra over a vector space E is $\text{FA}fA[E]$ together with the map δ .*

Proof. Being a sublattice-algebra of the Archimedean f -algebra $\mathbb{R}^{E^\#}$, $\text{FA}fA[E]$ is again an Archimedean f -algebra. Moreover, the map δ is linear. Indeed, if

$x, y \in E$, $\lambda, \mu \in \mathbb{R}$ and $\omega \in E^\#$, then

$$\delta_{\lambda x + \mu y}(\omega) = \omega(\lambda x + \mu y) = \lambda \omega(x) + \mu \omega(y) = (\lambda \delta_x + \mu \delta_y)(\omega).$$

Let A be an Archimedean f -algebra and let $T: E \rightarrow A$ be a linear map. For $f \in \text{FAfA}[E]$, say $f = \Phi(\delta_{x_1}, \dots, \delta_{x_n})$ for some LLA expression Φ and $x_1, \dots, x_n \in E$, define $\hat{T}(f) = \Phi(Tx_1, \dots, Tx_n)$. As in the proof of Theorem 8.8, the only non-trivial part is to check that $\hat{T}$ is well-defined. For this it suffices to check that, if $\Phi(\delta_{x_1}, \dots, \delta_{x_n}) = 0$ in $\text{FAfA}[E]$, then $\Phi(Tx_1, \dots, Tx_n) = 0$ in A .

Let $\{z_1, \dots, z_m\}$ be a basis of the span of $\{x_1, \dots, x_n\}$. Write, for $i = 1, \dots, n$,

$$x_i = \sum_{j=1}^m \alpha_{ij} z_j \quad \text{for certain } \alpha_{ij} \in \mathbb{R}.$$

Define the LLA expression

$$\Psi[t_1, \dots, t_m] = \Phi \left[\sum_{j=1}^m \alpha_{1j} t_j, \dots, \sum_{j=1}^m \alpha_{nj} t_j \right].$$

By the linearity of δ ,

$$0 = \Phi(\delta_{x_1}, \dots, \delta_{x_n}) = \Psi(\delta_{z_1}, \dots, \delta_{z_m}),$$

which implies

$$\Psi(\omega(z_1), \dots, \omega(z_m)) = 0$$

for every $\omega \in E^\#$. Since $z_1, \dots, z_m$ are linearly independent, $(\omega(z_1), \dots, \omega(z_m))$ ranges through $\mathbb{R}^m$ as ω ranges through $E^\#$. By Theorem 8.2, Ψ vanishes on A . Using the linearity of T it follows that

$$\Phi(Tx_1, \dots, Tx_n) = \Phi \left(\sum_{j=1}^m \alpha_{1j} Tz_j, \dots, \sum_{j=1}^m \alpha_{nj} Tz_j \right) = \Psi(Tz_1, \dots, Tz_m) = 0.$$

□

Remark 8.47. Note that, when E is a normed space, the previous construction can be carried out in $\mathbb{R}^{E^*}$ instead of $\mathbb{R}^{E^\#}$. In contrast with the free vector lattice over a vector space, however, we cannot go one step further and restrict the elements of $\text{FAfA}[E]$ to B_{E^*} so as to obtain a representation of $\text{FAfA}[E]$ in $C(B_{E^*}, w^*)$. The reason is that LL expressions are positively homogeneous, while LLA expressions need not be (for instance, in $\text{FAfA}[\mathbb{R}]$ the non-zero element $((\delta_1)_+^2 - (\delta_1)_+)_+$ vanishes on $B_{\mathbb{R}^*} = [-1, 1]$; see Remark 8.26). Despite this, it is remarkable that in Section 8.3.2 and Section 8.4 we will still be able to represent $\text{FNfA}[E]$ and, in some cases, $\text{FBfA}[E]$ inside $C(B_{E^*})$.

Next we proceed to the abstract construction of $\text{FNfA}[E]$ for a normed vector space E . In $\text{FAfA}[E]$ (regarded as the sublattice-algebra of $\mathbb{R}^{E^*}$ generated by $\{\delta_x : x \in E\}$), let $\mathcal{N}$ be the set of lattice seminorms ν that are submultiplicative and satisfy $\nu(\delta_x) \leq \|x\|$ for all $x \in E$. This set is not empty, for if $x^* \in B_{E^*}$, then $\nu_{x^*}(f) = |f(x^*)|$ defines an element of $\mathcal{N}$.

Let $\rho(f) = \sup_{\nu \in \mathcal{N}} \nu(f)$ for every $f \in \text{FA}fA[E]$. We claim that ρ defines a submultiplicative lattice seminorm on $\text{FA}fA[E]$. First, we need to show that the quantity $\rho(f)$ is actually finite. If $f = \Phi(\delta_{x_1}, \dots, \delta_{x_n})$, for a certain LLA expression Φ and $x_1, \dots, x_n \in E$, it is not difficult to check, by induction on the complexity of Φ , that there exists a polynomial $p(t_1, \dots, t_n) \in \mathbb{R}_+[t_1, \dots, t_n]$ with $p(0, \dots, 0) = 0$ such that

$$|f| \leq p(|\delta_{x_1}|, \dots, |\delta_{x_n}|). \quad (8.2)$$

Then for every $\nu \in \mathcal{N}$:

$$\nu(|f|) \leq \nu(p(|\delta_{x_1}|, \dots, |\delta_{x_n}|)) \leq p(\nu(|\delta_{x_1}|), \dots, \nu(|\delta_{x_n}|)) \leq p(\|x_1\|, \dots, \|x_n\|),$$

where in the second inequality we have used the triangle inequality and that ν is submultiplicative, and in both the second and the third inequalities that the coefficients of p are positive. So $\rho(f) = \sup_{\nu \in \mathcal{N}} \nu(f) \leq p(\|x_1\|, \dots, \|x_n\|)$ is indeed finite. That $\rho(f)$ is a submultiplicative lattice seminorm follows from the definition.

The kernel of this seminorm

$$\ker \rho = \{ f \in \text{FA}fA[E] : \rho(f) = 0 \}$$

is an order and algebraic ideal. An elementary argument shows that the quotient $\text{FA}fA[E] / \ker \rho$ is again an f -algebra (see [Bou02, Proposition 3.2]). Denote this quotient by $\text{FN}fA[E]$, denote by $q_E: \text{FA}fA[E] \rightarrow \text{FN}fA[E]$ the quotient map, and set $\eta_E = q_E \delta_E$. Then ρ induces a norm in $\text{FN}fA[E]$ that makes it into a normed f -algebra. Moreover, the map η_E is contractive with respect to this norm (in fact, it is an isometry; see below). Denote the norm simply by $\|\cdot\|$ and let $\text{FB}fA[E]$ be the completion of $\text{FN}fA[E]$ with respect to this norm. Abusing the notation, we shall denote also by η_E the composition of this map with the canonical embedding $\text{FN}fA[E] \rightarrow \text{FB}fA[E]$.

Proposition 8.48. *The free normed f -algebra generated by a normed space E is $\text{FN}fA[E]$ together with the map η_E . The free Banach f -algebra generated by a Banach space E is $\text{FB}fA[E]$ together with the map η_E .*

Proof. We need to check that η_E is a linear isometric embedding and that $\text{FN}fA[E]$, together with η_E , satisfies the universal property defining the free normed f -algebra generated by E . Let $(A, \|\cdot\|_A)$ be a normed f -algebra, and let $T: E \rightarrow A$ be a contractive operator. Let $\tilde{T}: \text{FA}fA[E] \rightarrow A$ be the unique lattice-algebra homomorphism satisfying $\tilde{T} \delta_E = T$. It is direct to check that, since $\|T\| \leq 1$, $f \mapsto \|\tilde{T}f\|_A / \|T\|$ defines an element of $\mathcal{N}$. This implies that $\|\tilde{T}f\|_A \leq \|T\| \rho(f)$ for all $f \in \text{FA}fA[E]$. Hence there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FA}fA[E] / \ker \rho \rightarrow A$ satisfying $\hat{T} q_E = \tilde{T}$ and $\|\hat{T}\| \leq \|T\|$. This implies $\hat{T} \eta_E = \hat{T} q_E \delta_E = \tilde{T} \delta_E = T$ and, since η_E is contractive, also $\|\hat{T}\| = \|T\|$.

It only remains to show that η_E is an isometric embedding. Let $T: E \rightarrow \ell_\infty(B_{E^*})$ be the standard isometric embedding $Tx = (x^*(x))_{x^* \in B_{E^*}}$. Since

$\ell_\infty(B_{E^*})$, with coordinatewise order and product, is a Banach f -algebra, there exists a lattice-algebra homomorphism $\hat{T}: \text{FNfA}[E] \rightarrow \ell_\infty(B_{E^*})$ satisfying $\|\hat{T}\| = 1$ and $\hat{T}\eta_E = T$. Using that η_E is contractive, it follows that

$$\|x\| = \|Tx\| \leq \|\hat{T}\| \|\eta_E x\| = \|\eta_E x\| \leq \|x\|$$

holds for all $x \in E$. Therefore $\|\eta_E x\| = \|x\|$.

It is direct that the completion of $\text{FNfA}[E]$, namely $\text{FBfA}[E]$, together with η_E , is the free Banach f -algebra generated by E . $\square$

Remark 8.49. Let Z be the closure of $\text{Lat}\{\eta_x : x \in E\}$ in $\text{FBfA}[E]$. Let $\phi_E: E \rightarrow Z$ be η_E with the codomain restricted to Z . Then Z , together with ϕ_E , is the free Banach lattice $\text{FBL}[E]$. Indeed, let X be a Banach lattice, and let $T: E \rightarrow X$ be an operator. Equip X with the zero product, so that it becomes a Banach f -algebra. Then $S = T/\|T\|$ extends to a lattice-algebra homomorphism $\hat{S}: \text{FBfA}[E] \rightarrow X_0$ satisfying $\hat{S}\eta_E = S$ and $\|\hat{S}\| = 1$. Let $\hat{T}: Z \rightarrow X$ be the restriction of the lattice homomorphism $\|T\|\hat{S}$ to Z . Then $\hat{T}$ is a lattice homomorphism satisfying $\hat{T}\phi_E = T$. In particular, $\|T\| \leq \|\hat{T}\| \leq \|T\|\|\hat{S}\| = \|T\|$, so $\|\hat{T}\| = \|T\|$. An immediate consequence of this is that the free norm in $\text{FBfA}[E]$, when restricted to Z , is the norm of the free Banach lattice, for which an explicit formula was found in [ART18]. From now on, Z will be denoted as $\text{FBL}[E]$.

Moreover, $\text{FBL}[E]$ is contractively complemented in $\text{FBfA}[E]$ by a lattice homomorphism. Indeed, the canonical inclusion $\eta_E: E \rightarrow \text{FBL}[E]_0$ extends to a unique contractive lattice-algebra homomorphism $P: \text{FBfA}[E] \rightarrow \text{FBL}[E]_0$. Forget about the zero product in $\text{FBL}[E]$, and consider P merely as a lattice homomorphism. The restriction $P|_{\text{FBL}[E]}: \text{FBL}[E] \rightarrow \text{FBL}[E]$ is a lattice homomorphism of norm one satisfying $P\eta_E = \eta_E$. By uniqueness, $P|_{\text{FBL}[E]}$ is the identity on $\text{FBL}[E]$. Hence the composition of P with the canonical inclusion $\text{FBL}[E] \rightarrow \text{FBfA}[E]$ is a projection of $\text{FBfA}[E]$ onto $\text{FBL}[E]$.

8.3.2 Explicit description of the kernel

Thus far we have shown, using only general arguments, that $\text{FNfA}[E]$ and $\text{FBfA}[E]$ exist for a general Banach space E . But since we have quotiented out $\text{FAfA}[E]$ (which we know explicitly) by a mysterious ideal $\ker \rho$, we have no explicit description for $\text{FNfA}[E]$. Fortunately, we can use Theorem 8.25 to describe this ideal.

Theorem 8.50. *Let E be a Banach space. Let ρ be the greatest submultiplicative lattice seminorm on $\text{FAfA}[E]$ such that $\rho(\delta_x) \leq \|x\|$ for all $x \in E$. Then*

$$\ker \rho = \{f \in \text{FAfA}[E] : f|_{B_{E^*}} = 0\}.$$

Proof. For every $x^* \in B_{E^*}$, $\nu_{x^*}(f) = |f(x^*)|$ defines a lattice seminorm that is submultiplicative and such that $\nu_{x^*}(\delta_x) \leq \|x\|$ for all $x \in E$. If $f \in \ker \rho$, then

$$|f(x^*)| = \nu_{x^*}(f) \leq \rho(f) = 0.$$

Hence every function in $\ker \rho$ vanishes on the unit ball.

Conversely, suppose $f \in \text{FAfA}[E]$ is such that $f|_{B_{E^*}} = 0$. Write $f = \Phi(\delta_{x_1}, \dots, \delta_{x_n})$ for a certain LLA expression Φ and some $x_1, \dots, x_n \in E$; by changing Φ as necessary, we may assume that $x_1, \dots, x_n$ are linearly independent and $\|x_i\| \leq 1$ (see the proof of Proposition 8.46). The goal is to show that

$$q_E f = \Phi(q_E \delta_{x_1}, \dots, q_E \delta_{x_n}) = \Phi(\eta_E(x_1), \dots, \eta_E(x_n))$$

is zero in $\text{FNfA}[E]$. By Theorem 8.20 there exists an injective and contractive lattice-algebra homomorphism $R: \text{FNfA}[E] \rightarrow X_0 \oplus_\infty C(K)$, for a certain Banach lattice X and a certain compact Hausdorff space K . Denote by R_1 (resp. R_2) the composition of R with the projection onto the first (resp. second) coordinate. We will show that $R_i(q_E f) = 0$ for $i = 1, 2$.

For every $x^* \in B_{E^*}$:

$$0 = f(x^*) = \Phi(x^*(x_1), \dots, x^*(x_n)).$$

We claim that there exists an $\varepsilon > 0$ such that $(x^*(x_1), \dots, x^*(x_n))$ takes all values in $[-\varepsilon, \varepsilon]^n$ as x^* ranges through B_{E^*} . Indeed, set $F = \text{span}\{x_1, \dots, x_n\}$, and let $\|\cdot\|_1$ denote the ℓ_1^n -norm in F . Since F is finite-dimensional, there exists $C > 0$ such that $\|\cdot\|_1 \leq C\|\cdot\|$, where $\|\cdot\|$ stands for the norm in E . Define $\varepsilon = 1/C$, and let $x^*: F \rightarrow \mathbb{R}$ be a linear functional satisfying $(x^*(x_1), \dots, x^*(x_n)) \in [-\varepsilon, \varepsilon]^n$. If $x = \sum_{i=1}^n \lambda_i x_i$ is an arbitrary element of F , then

$$|x^*(x)| \leq \sum_{i=1}^n |\lambda_i| |x^*(x_i)| \leq \varepsilon \|x\|_1 \leq \|x\|.$$

This shows that $x^* \in B_{F^*}$. Use Hahn–Banach to extend this functional to an element $x^* \in B_{E^*}$. Since $(x^*(x_1), \dots, x^*(x_n))$ can be chosen arbitrarily in $[-\varepsilon, \varepsilon]^n$, the desired claim follows. It is a consequence of this claim and Lemma 8.7 that Φ vanishes on every Banach lattice with the zero product; in particular, it vanishes on X_0 , and therefore $R_1(q_E(f)) = 0$.

Since R_2 is a contractive lattice-algebra homomorphism, $R_2 = \hat{T}$ for a certain contractive operator $T: E \rightarrow C(K)$ satisfying $R_2 \eta_E = T$. For $t \in K$, denote by $\phi_t \in C(K)^*$ the evaluation functional at t . Recall that ϕ_t is a lattice-algebra homomorphism of norm one. Compute:

$$\begin{aligned} R_2(q_E(f))(t) &= \Phi(R_2(\eta_E(x_1))(t), \dots, R_2(\eta_E(x_n))(t)) \\ &= \Phi((Tx_1)(t), \dots, (Tx_n)(t)) \\ &= \Phi((T^* \phi_t)(x_1), \dots, (T^* \phi_t)(x_n)) \\ &= \Phi(\delta_{x_1}(T^* \phi_t), \dots, \delta_{x_n}(T^* \phi_t)) \\ &= \Phi(\delta_{x_1}, \dots, \delta_{x_n})(T^* \phi_t) \\ &= f(T^* \phi_t) = 0 \end{aligned}$$

since $\|T^* \phi_t\| \leq 1$ and f vanishes on the unit ball. This shows that $R_2(q_E(f)) = 0$. Hence $R(q_E(f)) = 0$ and, since R is injective, $q_E(f) = 0$. $\square$

Since $\delta_x: E^* \rightarrow \mathbb{R}$ is weak* continuous, so is every element of $\text{FA}fA[E]$. Therefore there is a restriction map:

$$\begin{array}{ccc} \text{FA}fA[E] & \longrightarrow & C(B_{E^*}, w^*) \\ f & \longmapsto & f|_{B_{E^*}} \end{array}.$$

Operations in $\text{FA}fA[E]$ are computed pointwise, so this is a lattice-algebra homomorphism. However, as noted in Remark 8.47, this map is not injective, so we cannot view the elements of $\text{FA}fA[E]$ as continuous functions on the unit ball of the dual.

It is remarkable that this map does induce an injective lattice-algebra homomorphism $\iota: \text{FN}fA[E] \rightarrow C(B_{E^*})$ because, according to Theorem 8.50, its kernel is precisely $\ker \rho$. Note that, by definition of the norm in $\text{FN}fA[E]$, ι is contractive. We have thus found an explicit lattice-algebraic representation of $\text{FN}fA[E]$.

Corollary 8.51. *Let E be a Banach space. There exists a unique injective and contractive lattice-algebra homomorphism*

$$\iota_E: \text{FN}fA[E] \rightarrow C(B_{E^*})$$

satisfying $(\iota_E \eta_x)(x^*) = x^*(x)$ for all $x \in E$ and $x^* \in B_{E^*}$.

From now on, ι_E (or simply ι when E is clear from the context) will denote the map in the above corollary. The following is a first application of this representation result.

Proposition 8.52. *Let E be a Banach space and let $f \in \text{FN}fA[E]$. There exist $\lambda > 0$ and $x_1, \dots, x_n \in E$ such that*

$$|f| \leq \lambda(|\eta_{x_1}| \vee \dots \vee |\eta_{x_n}|).$$

Proof. As already mentioned in Section 8.3.1, there exists a polynomial $p(t_1, \dots, t_n) \in \mathbb{R}_+[t_1, \dots, t_n]$ with $p(0, \dots, 0) = 0$ such that $|f| \leq p(|\eta_{x_1}|, \dots, |\eta_{x_n}|)$. Without loss of generality, $x_1, \dots, x_n \in B_E$. It follows that for every $x^* \in B_{E^*}$:

$$\begin{aligned} \iota(|f|)(x^*) &\leq p(|x^*(x_1)|, \dots, |x^*(x_n)|) \\ &\leq p(1, \dots, 1) |x^*(x_1)| \vee \dots \vee |x^*(x_n)| \\ &= p(1, \dots, 1) \iota(|\eta_{x_1}| \vee \dots \vee |\eta_{x_n}|)(x^*), \end{aligned}$$

where in the second inequality we have used the fact that $|x^*(x_i)| \leq 1$. Since ι is an injective lattice homomorphism, the result follows. $\square$

Remark 8.53. Since, a priori, $\iota: \text{FN}fA[E] \rightarrow C(B_{E^*})$ is not an embedding (in fact, we will show in Remark 8.61 that it is never an embedding), it is not clear at all whether its extension to $\text{FB}fA[E]$ is injective. In Section 8.4 we will use a completely different technique to show that the extension is indeed injective for some classes of Banach spaces.

Remark 8.54. Recall from Remark 8.49 that $\text{FBL}[E]$ can be seen as a closed sublattice of $\text{FBfA}[E]$; moreover, $\text{FBL}[E]$ is 1-complemented by a lattice homomorphic projection P . Using the representation of $\text{FNfA}[E]$ inside $C(B_{E^*})$, we can now give an explicit expression for P on $\text{FNfA}[E]$. We claim that, for every $f \in \text{FNfA}[E]$ and every $x^* \in B_{E^*}$,

$$\iota(Pf)(x^*) = \lim_{t \rightarrow 0^+} \frac{\iota(f)(tx^*)}{t}.$$

Indeed, consider the map $f \mapsto Pf$ defined by the above formula. According to Lemma 8.6 the limit exists, and $\iota(Pf)$ belongs to $\iota(\text{Lat}\{\eta_x : x \in E\})$; that is, Pf belongs to $\text{FVL}[E] \subseteq \text{FBL}[E]$ (see Remark 8.15 for further details). The operator P , as defined by the formula above, is a lattice-algebra homomorphism $\text{FNfA}[E] \rightarrow \text{FBL}[E]_0$ satisfying $P\eta_x = \eta_x$. Therefore, P is the unique extension of $\eta : E \rightarrow \text{FBL}[E]_0$ to $\text{FNfA}[E]$. The extension of P by density to the whole $\text{FBfA}[E]$ must be, again by uniqueness, the contractive lattice projection described in Remark 8.49.

Proposition 8.55. *The free normed f -algebra $\text{FNfA}[E]$ generated by a Banach space E is semiprime.*

Proof. We will use the representation $\iota : \text{FNfA}[E] \rightarrow C(B_{E^*})$. Let $f \in N(\text{FNfA}[E])$. For every $x^* \in B_{E^*}$ different from 0 there exists an $x \in B_E$ such that $\iota(\eta_x)(x^*) = x^*(x) \neq 0$. Then

$$0 = \iota(f\eta_x)(x^*) = \iota(f)(x^*)x^*(x).$$

It follows that $\iota(f)(x^*) = 0$. Since also $\iota(f)(0) = 0$, it must be $\iota(f) = 0$. But ι is injective, so $f = 0$. $\square$

Although the lattice-algebraic representation of $\text{FNfA}[E]$ provided in Corollary 8.51 has proven useful, not much else can be said about free normed f -algebras unless we investigate the properties of its norm. The following section is a first step in this direction: it shows in particular that, when E is finite-dimensional, the free norm is equivalent to an AM-norm.

8.3.3 The finite-dimensional case

The goal of this section is to prove the following isomorphic description of the space $\text{FBfA}[E]$ when E is a finite-dimensional Banach space.

Theorem 8.56. *Let E be a finite-dimensional Banach space. The free Banach f -algebra $\text{FBfA}[E]$ is lattice-algebra isomorphic to the space $C([0, 1] \times S_{E^*})$ equipped with pointwise order and product*

$$(f \star g)(r, u) = rf(r, u)g(r, u) \tag{8.3}$$

where $(r, u) \in [0, 1] \times S_{E^*}$ and $f, g \in C([0, 1] \times S_{E^*})$. Under this isomorphism, $\eta_x(r, u) = u(x)$ for every $x \in E$.

Figure 8.1 illustrates the generators and the “weight” $\mathbb{1}_K \star \mathbb{1}_K$ (where $\mathbb{1}_K$ denotes the constant one function on $K = [0, 1] \times S_{\ell_2^2}$) of the free Banach f -algebra generated by ℓ_2^2 .

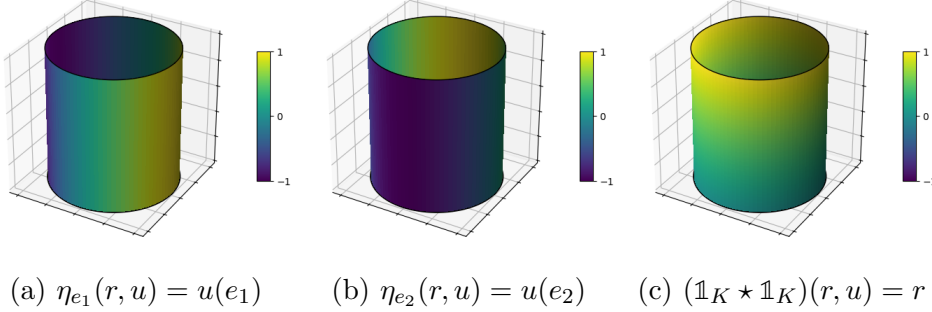


Figure 8.1: Representation of three functions in $C([0, 1] \times S_{\ell_2^2})$. In the plots, $[0, 1]$ is identified with $0 \leq z \leq 1$ and $S_{\ell_2^2}$ is identified with the unit circle in the plane $z = 0$. The elements e_1, e_2 denote the canonical basis of ℓ_2^2 .

To make the steps in the proof clearer, we have divided it into several lemmas. First, we want to show that there exists an element in $\text{FNfA}[E]$ that is a strong unit in $\text{FBfA}[E]$, and that the norm it induces is equivalent to the free norm.

Lemma 8.57. *Let E be a finite-dimensional Banach space. There exists a finite subset $F \subseteq S_E$ for which $e' = \sup_{x \in F} |\eta_x|$ is a strong unit in $\text{FBfA}[E]$ and such that the norm it induces $\|\cdot\|_{e'}$ is equivalent to the free norm.*

Proof. We shall first show that $\text{FBfA}[E]$ has a strong unit $e \in \text{FBfA}[E]$. Denote by $\text{Fin}(S_E)$ the family of finite subsets of S_E . For every $F \in \text{Fin}(S_E)$, define the element $e_F = \sup_{x \in F} |\eta_x| \in \text{FNfA}[E]$. In fact, this element belongs to $\text{FVL}[E]$, where we are identifying $\text{FVL}[E]$ inside $\text{FNfA}[E]$ as in Remark 8.49. Keep in mind that, with this identification, the norm of $\text{FNfA}[E]$, when restricted to $\text{FVL}[E]$, is precisely the norm of the free Banach lattice. Hence we can refer to the free norm on $\text{FVL}[E]$ without ambiguity.

Let $\iota: \text{FNfA}[E] \rightarrow C(B_{E^*})$ be the injective lattice-algebra homomorphism from Corollary 8.51. For every $x^* \in B_{E^*}$, the net $(\iota(e_F)(x^*))_{F \in \text{Fin}(S_E)}$ converges to $\|x^*\|_{E^*}$. Since E is finite-dimensional, $\|\cdot\|_{E^*}$ is a continuous function on B_{E^*} (with respect to the weak* topology). Moreover, $(e_F)_{F \in \text{Fin}(S_E)}$ is an increasing net. By Dini’s theorem, $\iota(e_F)$ converges uniformly to $\|\cdot\|_{E^*}$ in $C(B_{E^*})$.

Recall from [OTTT24, Section 9.1] that, because E is finite-dimensional, the free norm is equivalent to the uniform norm on $\iota(\text{FVL}[E])$; that is, ι is an embedding when restricted to $\text{FVL}[E]$. We have shown above that the net $(\iota(e_F))$ is Cauchy in the uniform norm. Hence the net (e_F) is Cauchy in the free norm.

Let $e \in \text{FBfA}[E]$ be the limit of (e_F) . Since (e_F) is an increasing net, $e = \sup_{F \in \text{Fin}(S_E)} e_F$. We are going to show that I_e , the order ideal generated by e , is the whole $\text{FBfA}[E]$ (and therefore that e is a strong unit). For this, we

first need to show that the free norm $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent. It is clear that $\iota(e_F)^2 \leq \iota(e_F)$ holds pointwise in B_{E^*} for every $F \in \text{Fin}(S_E)$. Hence $e_F^2 \leq e_F$ and, taking limits, $e^2 \leq e$. It follows that, for every $f, g \in I_e$,

$$|fg| \leq |f||g| \leq \|f\|_e \|g\|_e e^2 \leq \|f\|_e \|g\|_e e.$$

Therefore $fg \in I_e$ and $\|fg\|_e \leq \|f\|_e \|g\|_e$. Also note that, if $x \in S_E$, then $|\eta_x| = e_{\{x\}} \leq e$. In particular, $\text{FNfA}[E] \subseteq I_e$ and $\|\eta_x\|_e \leq \|x\|$ for every $x \in E$. We have thus shown that $\|\cdot\|_e$, when restricted to $\text{FNfA}[E]$, is a submultiplicative lattice norm satisfying $\|\eta_x\|_e \leq \|x\|$ for all $x \in E$. By construction of the free norm, $\|\cdot\|_e \leq \|\cdot\|$ on $\text{FNfA}[E]$.

On the other hand, $|f| \leq \|f\|_e e$ holds for every $f \in \text{FNfA}[E]$. By taking the free norm on both sides of the inequality, $\|f\| \leq \|f\|_e \|e\|$. This completes the proof that the free norm is equivalent to $\|\cdot\|_e$ on $\text{FNfA}[E]$. Now if $f \in \text{FBfA}[E]$ is arbitrary, and $f_n \rightarrow f$ in the free norm, with $f_n \in \text{FNfA}[E]$, then (f_n) is also a Cauchy sequence in the norm $\|\cdot\|_e$. Since this norm is complete, (f_n) converges to an element of I_e ; since this norm is equivalent to the free norm, the limit must be f . Hence, $f \in I_e$. This shows that e is a strong unit, and that the norm $\|\cdot\|_e$ is equivalent to the free norm on $\text{FBfA}[E]$.

To finish the proof, it only remains to show that there exists a strong unit $e' \in \text{FNfA}[E]$ of the desired form equivalent to e . For this, it suffices to find $e' \in \text{FNfA}[E]$ of the desired form satisfying $e/2 \leq e' \leq e$. (The factor $1/2$ does not play any special role in the proof; any scalar in $(0, 1)$ would also work.) Let $F \in \text{Fin}(S_E)$ be such that

$$\|\iota(e_F) - \|\cdot\|_{E^*}\|_\infty \leq 1/2$$

and put $e' = e_F$. Certainly, $e' \leq e$. For every $u \in S_{E^*}$, $|\iota(e')(u) - 1| \leq 1/2$, which implies $\iota(e')(u) \geq 1/2$. Since $\iota(e')$ is positively homogeneous, it follows that

$$\iota(e')(x^*) \geq 1/2 \|x^*\|_{E^*} \geq \iota(e_{F'}/2)(x^*)$$

for all $x^* \in B_{E^*}$ and all $F' \in \text{Fin}(S_E)$. Hence $e' \geq e_{F'}/2$ in $\text{FBfA}[E]$ for all $F' \in \text{Fin}(S_E)$. Taking the limit in $F' \in \text{Fin}(S_E)$, it follows that $e' \geq e/2$. $\square$

Remark 8.58. The subset F in the previous lemma is far from unique. In fact, if $F \subseteq F'$, then F' also satisfies the desired properties.

From the previous lemma and Kakutani's theorem it already follows that $\text{FBfA}[E]$ is lattice isomorphic to $C(K)$, for a certain compact Hausdorff space K .

Lemma 8.59. *Let E be a finite-dimensional Banach space. Let $C([0, 1] \times S_{E^*})$ be equipped with pointwise order and product $\star$ given by (8.3). Then the map $T: E \rightarrow C([0, 1] \times S_{E^*})$, defined by*

$$(Tx)(r, u) = u(x) \quad \text{for all } x \in E, (r, u) \in [0, 1] \times S_{E^*},$$

extends to a contractive lattice-algebra homomorphism $\hat{T}: \text{FBfA}[E] \rightarrow C([0, 1] \times S_{E^*})$ whose explicit expression is

$$(\hat{T}f)(r, u) = \begin{cases} \frac{\iota(f)(ru)}{r} & \text{if } r \neq 0 \\ \lim_{r \rightarrow 0^+} \frac{\iota(f)(ru)}{r} & \text{if } r = 0 \end{cases}$$

for every $f \in \text{FNfA}[E]$.

Proof. It is direct to check that $C([0, 1] \times S_{E^*})$ with the usual lattice structure, the product $\star$, and the uniform norm, is a Banach f -algebra. Note that

$$\|Tx\|_\infty = \sup_{u \in S_{E^*}} |u(x)| = \|x\|.$$

Hence T extends in a unique way to a norm-one lattice-algebra homomorphism $\hat{T}: \text{FBfA}[E] \rightarrow C([0, 1] \times S_{E^*})$.

Denote momentarily

$$Sf(r, u) = \begin{cases} \frac{\iota(f)(ru)}{r} & \text{if } r \neq 0 \\ \lim_{r \rightarrow 0^+} \frac{\iota(f)(ru)}{r} & \text{if } r = 0 \end{cases}$$

for every $(r, u) \in [0, 1] \times S_{E^*}$ and $f \in \text{FNfA}[E]$. To show that $\hat{T}|_{\text{FNfA}[E]} = S$, we will check that the map $S: \text{FNfA}[E] \rightarrow C([0, 1] \times S_{E^*})$, $f \mapsto Sf$, is a lattice-algebra homomorphism extending T . Uniqueness of the extension will then imply that $\hat{T}|_{\text{FNfA}[E]} = S$.

First of all, note that the limit defining $Sf(0, u)$ exists by Lemma 8.6. We need to check that Sf is continuous on $[0, 1] \times S_{E^*}$. It is clear that it is continuous on $\{(r, u) \in [0, 1] \times S_{E^*} : r \neq 0\}$. Let $(r_n, u_n) \in [0, 1] \times S_{E^*}$ be a sequence converging to a certain $(0, u) \in [0, 1] \times S_{E^*}$. The limit defining $Sf(0, u)$ coincides with $f_0(u) = \lim_n f(r_n u) r_n^{-1}$ for every $u \in S_{E^*}$. Since E is finite-dimensional, S_{E^*} is a compact metric space under the metric induced by the norm. Moreover, S_{E^*} can be embedded isometrically as a compact subspace of $\mathbb{R}^n$. Then f_0 is, by Lemma 8.6, the uniform limit of continuous functions. It is, therefore, continuous. Furthermore, since S_{E^*} is compact, f_0 is uniformly continuous on S_{E^*} .

Fix $\varepsilon > 0$. Choose $\delta > 0$ such that $|f_0(u) - f_0(u')| < \varepsilon/2$ whenever $\|u - u'\|_{E^*} < \delta$ and such that $|f(r_n u) r_n^{-1} - f_0(u)| < \varepsilon/2$ whenever $0 \leq r_n < \delta$ and $u \in S_{E^*}$ (recall the limit defining $f_0(u)$ is uniform). For $n \in \mathbb{N}$ big enough we will have $r_n, \|u - u_n\| < \delta$, and then, whenever $r_n \neq 0$,

$$\begin{aligned} |Sf(r_n, u_n) - f_0(u)| &= |f(r_n u_n) r_n^{-1} - f_0(u)| \\ &\leq |f(r_n u_n) r_n^{-1} - f_0(u_n)| + |f_0(u_n) - f_0(u)| < \varepsilon. \end{aligned}$$

When $r_n = 0$, $Sf(r_n, u_n) = f_0(u_n)$ and $|f_0(u_n) - f_0(u)| < \varepsilon/2$. This proves that $Sf(r_n, u_n) \rightarrow f_0(u) = Sf(0, u)$, and therefore that Sf is a continuous function.

It is clear from the definition that S is a lattice homomorphism. It is also an algebra homomorphism because, for $r \neq 0$,

$$S(fg)(r, u) = \iota(fg)(ru)r^{-1} = rSf(r, u)Sg(r, u) = (Sf \star Sg)(r, u).$$

The equality also holds for general $(r, u) \in [0, 1] \times S_{E^*}$ by continuity. Certainly, $(S\eta_x)(r, u) = u(x) = (Tx)(r, u)$. Hence $Sf = \hat{T}f$ for all $f \in \text{FNfA}[E]$. $\square$

With the explicit expression we can now show that the map $\hat{T}$ is injective on $\text{FNfA}[E]$.

Lemma 8.60. *Let E be a finite-dimensional Banach space, and let $\hat{T}$ be as in Lemma 8.59. If, for some $f, g \in \text{FNfA}[E]$, $\hat{T}f \leq \hat{T}g$, then $f \leq g$.*

Proof. Let us first check that $\hat{T}$ is injective on $\text{FNfA}[E]$. Suppose $\hat{T}f = 0$ for some $f \in \text{FNfA}[E]$. Then, for every $u \in S_{E^*}$ and $0 < r \leq 1$,

$$0 = (\hat{T}f)(r, u) = \iota(f)(ru)/r.$$

This implies that $\iota(f)$ vanishes on $B_{E^*} \setminus \{0\}$. Since every element of $\text{FNfA}[E]$ vanishes also at 0, it follows that $\iota(f) = 0$. Injectivity of ι implies that $f = 0$. Hence the map $\hat{T}|_{\text{FNfA}[E]}: \text{FNfA}[E] \rightarrow \hat{T}(\text{FNfA}[E])$ is a lattice isomorphism. In particular, if $\hat{T}f \leq \hat{T}g$, then by applying the inverse it follows that $f \leq g$. $\square$

Proof of Theorem 8.56. Let $e' = \sup_{x \in F} |\eta_x| \in \text{FNfA}[E]$ be as in Lemma 8.57, where $F \in \text{Fin}(S_E)$, and let $\hat{T}: \text{FBfA}[E] \rightarrow C([0, 1] \times S_{E^*})$ be as in Lemma 8.59. First note that

$$(\hat{T}e')(r, u) = \sup_{x \in F} |u(x)| \geq 1/2$$

(see the end of the proof of Lemma 8.57). Therefore $\hat{T}e'$ is a strong unit in $C([0, 1] \times S_{E^*})$. We are going to show that $\|\hat{T}f\|_{\hat{T}e'} = \|f\|_{e'}$ for all $f \in \text{FNfA}[E]$.

Since $|f| \leq \|f\|_{e'}e'$, applying $\hat{T}$ to this inequality yields immediately $\|\hat{T}f\|_{\hat{T}e'} \leq \|f\|_{e'}$. Similarly, $\hat{T}|f| = |\hat{T}f| \leq \|\hat{T}f\|_{\hat{T}e'}\hat{T}e'$ and from Lemma 8.60 it follows that $|f| \leq \|\hat{T}f\|_{\hat{T}e'}e'$. Hence also $\|f\|_{e'} \leq \|\hat{T}f\|_{\hat{T}e'}$.

Since the free norm is equivalent to $\|\cdot\|_{e'}$ and the uniform norm is equivalent to $\|\cdot\|_{\hat{T}e'}$, the equality $\|\hat{T}f\|_{\hat{T}e'} = \|f\|_{e'}$ must hold for all $f \in \text{FBfA}[E]$. In particular, $\hat{T}$ is injective and its image is closed under the uniform norm.

To complete the proof it only remains to show that $\hat{T}$ is surjective. This will follow as a consequence of the Stone–Weierstrass theorem once we show that the image of $\hat{T}$ contains the constant one function and separates points.

Define, as in the proof of Lemma 8.57, the element $e_F = \sup_{x \in F} |\eta_x|$ for every $F \in \text{Fin}(S_E)$. We already showed in the lemma that (e_F) converges to a certain $e \in \text{FBfA}[E]$ in the free norm. Note that

$$(\hat{T}e_F)(r, u) = \sup_{x \in F} |Tx|(r, u) = \sup_{x \in F} |u(x)|$$

converges to 1. Therefore $\hat{T}e = \lim_F \hat{T}e_F = \mathbb{1}_{[0, 1] \times S_{E^*}}$, the constant one function on $[0, 1] \times S_{E^*}$.

Let $(r, u), (r', u') \in [0, 1] \times S_{E^*}$. If $u \neq u'$, then there exists $x \in S_E$ such that $u(x) \neq u'(x)$. In other words:

$$(\hat{T}\eta_x)(r, u) = u(x) \neq u'(x) = (\hat{T}\eta_x)(r', u').$$

If instead $r \neq r'$, then we have

$$(\hat{T}e^2)(r, u) = (\hat{T}e \star \hat{T}e)(r, u) = r$$

and therefore $(\hat{T}e^2)(r, u) \neq (\hat{T}e^2)(r', u')$. Thus the image of $\hat{T}$ separates points of $[0, 1] \times S_{E^*}$. From the Stone–Weierstrass theorem it follows that $\hat{T}$ is surjective. $\square$

Remark 8.61. The representation $\iota: \text{FNfA}[E] \rightarrow C(B_{E^*})$ is never an embedding. Indeed, when E is infinite-dimensional, the norm of the free Banach lattice cannot be equivalent to an AM-norm (this is a consequence of [OTT24, Proposition 9.30]). Suppose that ι were an embedding for some finite-dimensional E ; we will show that this contradicts Theorem 8.56. In this case, the image of ι would be a closed sublattice-algebra of J_0 , the closed (lattice and algebraic) ideal of $C(B_{E^*})$ formed by the functions that vanish at 0. It is immediate to check that $\iota(\text{FNfA}[E])$ separates the points of B_{E^*} , and that for every $x^* \in B_{E^*}$ different from zero there exists $f \in \text{FNfA}[E]$ such that $\iota(f)(x^*) \neq 0$. As a consequence of the Stone–Weierstrass theorem, $\iota(\text{FNfA}[E]) = J_0$. But J_0 does not have a strong unit; this contradicts Theorem 8.56.

As we shall see in Section 8.4, determining whether $\text{FBfA}[E]$ is semiprime or not is very important and yet far from trivial.

Corollary 8.62. *If E is a finite-dimensional Banach space, then $\text{FBfA}[E]$ is semiprime.*

Proof. By Theorem 8.56, we only need to check that the product (8.3) is semiprime. Suppose $f \star f = 0$. Then

$$0 = (f \star f)(r, u) = rf(r, u)^2$$

for all $(r, u) \in [0, 1] \times S_{E^*}$. This means that f vanishes on the dense set $\{(r, u) \in [0, 1] \times S_{E^*} : r \neq 0\}$. By continuity, $f = 0$. $\square$

In the next result, S_∞^n denotes the unit sphere of ℓ_∞^n .

Corollary 8.63. *Let E be a Banach space of dimension n . The free Banach f -algebra $\text{FBfA}[E]$ is lattice-algebra isomorphic to $C([0, 1] \times S_\infty^n)$ with pointwise order, and product*

$$(f \star g)(r, u) = rf(r, u)g(r, u)$$

where $(r, u) \in [0, 1] \times S_\infty^n$ and $f, g \in C([0, 1] \times S_\infty^n)$.

Proof. Identify, by taking coordinates, S_{E^*} and S_∞^n with their homeomorphic copies in $\mathbb{R}^n$. The map

$$\begin{aligned} \phi: [0, 1] \times S_{E^*} &\longrightarrow [0, 1] \times S_\infty^n \\ (r, u) &\longmapsto (r, u/\|u\|_\infty) \end{aligned}$$

is a homeomorphism. Consider the associated composition operator

$$\begin{aligned} T: C([0, 1] \times S_\infty^n) &\longrightarrow C([0, 1] \times S_{E^*}) \\ f &\longmapsto f \circ \phi \end{aligned}.$$

This map is a lattice isometry. Moreover, if $f, g \in C([0, 1] \times S_\infty^n)$ and $(r, u) \in [0, 1] \times S_{E^*}$, then

$$\begin{aligned} T(f \star g)(r, u) &= (f \star g)(\phi(r, u)) \\ &= (f \star g)(r, u/\|u\|_\infty) \\ &= rf(r, u/\|u\|_\infty)g(r, u/\|u\|_\infty) \\ &= r(f \circ \phi)(r, u)(g \circ \phi)(r, u) \\ &= (Tf \star Tg)(r, u). \end{aligned}$$

This shows that T is also an algebra homomorphism. The stated result now follows from Theorem 8.56. $\square$

It might be pertinent to recall here that the free Banach lattice over a Banach space of dimension n is lattice isomorphic to $C(S_\infty^n)$, but the isomorphism constant increases without bound as the dimension grows (see [OTTT24, Section 9]).

Corollary 8.64. *Let E and F be Banach spaces of dimension n . Then $\text{FBfA}[E]$ and $\text{FBfA}[F]$ are lattice-algebra isomorphic.*

Remark 8.65. Even though isomorphic Banach spaces generate isomorphic free Banach lattices, this is not clear at all for free Banach f -algebras. In fact, in Example 8.93 we will provide an isomorphism between Banach spaces that does not induce an isomorphism between the corresponding free Banach f -algebras. It is clear that isometric Banach spaces generate lattice-algebra isometric free Banach f -algebras. The isomorphic case, however, remains open in infinite dimensions.

Question 8.66. Let E and F be isomorphic Banach spaces. Are $\text{FBfA}[E]$ and $\text{FBfA}[F]$ lattice-algebra isomorphic?

The converse of the previous question is open even for the free Banach lattice generated by a Banach space (see [OTTT24, Section 10]). The isometric version of the converse will be addressed in Section 8.5.4.

We have completely identified the free Banach f -algebra generated by a finite-dimensional Banach space, except for the free norm. The goal of the next section is to better understand this norm.

8.3.4 On the free norm

It follows from the abstract construction in Section 8.3.1 that the norm in $\text{FBfA}[E]$ is the supremum of the lattice seminorms that are submultiplicative and that make the canonical map $\eta_E: E \rightarrow \text{FBfA}[E]$ contractive. There is an equivalent way of defining this norm: for every $f \in \text{FBfA}[E]$, $\|f\|$ is the smallest number such that, whenever A is a Banach f -algebra and $T: E \rightarrow A$ is a contractive operator, $\|\hat{T}f\| \leq \|f\|$. Moreover, if $f \in \text{FBL}[E]$, then

$$\|f\| = \sup\{\|\hat{T}f\| : T: E \rightarrow \ell_1^n \text{ with } \|T\| \leq 1\}.$$

This important fact was first proved in [ART18] and is at the center of the theory of free Banach lattices. For this reason, in this section we try to reach a similar result for free Banach f -algebras. Only considering operators that arrive at ℓ_1^n is too ambitious; instead, we will take the supremum over all operators that arrive at finite-dimensional semiprime f -algebras. Although the semiprimeness condition may look artificial at first, it will turn out to be very useful in Section 8.4.

Definition 8.67. Let E be a Banach space. For every $f \in \text{FBfA}[E]$, define $\tau_E(f)$ to be the least positive number such that, if A is a semiprime finite-dimensional Banach f -algebra, and $T: E \rightarrow A$ is a contractive operator, then $\tau_E(f) \geq \|\hat{T}f\|$. When there is no ambiguity on the underlying Banach space, τ_E is simply denoted by τ .

Remark 8.68. Note that $\tau(f)$ is finite since $\|\hat{T}f\| \leq \|f\|$ for every contractive operator $T: E \rightarrow A$. In particular, $\tau(f) \leq \|f\|$. It is not difficult to check from the definition that $f \mapsto \tau(f)$ defines a submultiplicative lattice seminorm on $\text{FBfA}[E]$.

It will be very valuable for the discussion of Section 8.4 (and in particular for Proposition 8.80) to determine when τ defines a norm on $\text{FBfA}[E]$ (i.e., when $\tau(f) = 0$ implies $f = 0$). The following result is an important step in this direction.

Theorem 8.69. *The norm in $\text{FBfA}[\ell_1^n]$ is τ for every $n \in \mathbb{N}$.*

The main computation of the theorem is encapsulated in the next lemma. Since it is quite technical, let us explain the ideas that motivate it. From Theorem 8.56 we know that $\text{FBfA}[\ell_1^n]$ is isomorphic to $C(K)$, for a certain compact Hausdorff space K , with a product of the form $f \star g = wf g$, where $w \in C(K)_+$ and juxtaposition denotes the pointwise product. (We are not being explicit with the values of K and w on purpose; the argument will not depend on this.) Let $f = \Phi(\eta_{e_1}, \dots, \eta_{e_n}) \in \text{FBfA}[E]$, for a certain LLA expression Φ . Approximate the generators $\eta_{e_1}, \dots, \eta_{e_n}$ and the weight w by simple functions (i.e., functions that take only finitely many values). Disjointify the family of characteristic functions involved in these simple functions. Then their span will be a finite-dimensional sublattice A that contains discrete approximations of

$\eta_{e_1}, \dots, \eta_{e_n}$. Define a product in this space that consists in multiplying pointwise by the discrete approximation of w . If the approximations are good enough, this new product will be almost submultiplicative for the free norm, and Φ , when evaluated at the discrete approximations of the η_i , will be very close to f . This is the statement of the lemma. Afterwards, in the proof of the theorem, we will use this finite-dimensional f -algebra A , with an appropriate norm, to construct an operator $T: \ell_1^n \rightarrow A$ for which $\|\hat{T}f\|_A$ is very close to the free norm of f . One caveat though: since K is not sufficiently disconnected, we may not be able to approximate the functions by simple functions. For this reason we shall work in $B(K)$, the Banach lattice of bounded Borel-measurable functions on K .

Lemma 8.70. *Let K be a compact Hausdorff space, and let $C(K)$ be equipped with the usual lattice structure and a certain product $\star$ that makes it a Banach f -algebra. Let $f_1, \dots, f_n \in C(K)$, and let $f = \Phi(f_1, \dots, f_n)$ for some LLA expression Φ . Fix $\varepsilon, \varepsilon' > 0$. Then there exists a finite-dimensional sublattice A of $B(K)$, together with a semiprime f -algebra product $\circ$ in A , and $(f_1)_d, \dots, (f_n)_d \in A$ such that $|(f_i)_d| \leq |f_i|$ and, if $f_d = \Phi((f_1)_d, \dots, (f_n)_d)$ (where now Φ is evaluated using the product of A), then $\|f - f_d\|_\infty < \varepsilon$. Moreover, $\circ$ can be chosen so that*

$$|x \circ y| \leq |x \star y| + \varepsilon' \mathbf{1}_K^\dagger$$

holds for all $x, y \in A$ with $\|x\|_\infty, \|y\|_\infty \leq 1$.

Proof. Denote by $\star$ the product in $C(K)$. Since $C(K)$ with the product $\star$ is a Banach f -algebra, there exists a weight $w \in C(K)_+$, $\|w\|_\infty \leq 1$, such that $(f \star g)(t) = w(t)f(t)g(t)$ for all $f, g \in C(K)$ (Proposition 7.1).

Consider the bidual $C(K)^{**}$ equipped with the Arens product. This is again a Banach f -algebra, and we have a lattice-algebra isometry $C(K) \rightarrow C(K)^{**}$. Consider the Banach lattice $B(K)$ of bounded Borel-measurable functions on K . Note that $C(K) \subseteq B(K) \subseteq C(K)^{**}$ in a natural way, since every bounded Borel-measurable function can be integrated against an element of $M(K) = C(K)^*$. We are going to show that $B(K)$ is a subalgebra of $C(K)^{**}$, thus becoming a Banach f -algebra. Even better, the Arens product of $F, G \in B(K)$ is $(F \star G)(t) = w(t)F(t)G(t)$ because, if $f, g \in C(K)$ and $\mu \in M(K)$, then

$$\langle \mu f, g \rangle = \langle \mu, f \star g \rangle = \int_K g(t)w(t)f(t)d\mu(t)$$

implies $d(\mu f)(t) = w(t)f(t)d\mu(t)$, and

$$\langle G\mu, f \rangle = \langle G, \mu f \rangle = \int_K G(t)d(\mu f)(t) = \int_K f(t)w(t)G(t)d\mu(t)$$

implies $d(G\mu)(t) = w(t)G(t)d\mu(t)$. Therefore

$$\langle F \star G, \mu \rangle = \langle F, G\mu \rangle = \int_K F(t)w(t)G(t)d\mu(t)$$

[†]Here $\star$ denotes the Arens extension of the product in $C(K)$ to $B(K)$, see the proof below.

and, since this holds for all $\mu \in M(K)$, it follows that $(F \star G)(t) = w(t)F(t)G(t)$ for all $t \in K$.

Fix $\varepsilon > 0$. For convenience, we shall assume that $\|f_1\|_\infty, \dots, \|f_n\|_\infty \leq 1$; this can always be achieved by changing Φ if necessary. Let $0 = c_0 < c_1 < \dots < c_N < c_{N+1} = 1 + \delta$ be a partition of the interval $[0, 1 + \delta]$ with $\max_i |c_{i+1} - c_i| < \delta$, this $\delta > 0$ small enough and to be determined later. (We use $1 + \delta$ instead of 1 to force $c_N > 1$, since our functions can take the value 1.) Consider the algebra $\mathcal{C}$ of subsets of K generated by

$$\{c_i \leq (f_s)_\sigma < c_{i+1}\} \text{ for } s = 1, \dots, n, i = 0, \dots, N, \sigma \in \{+, -\} \text{ and}$$

$$\{c_i \leq w < c_{i+1}\} \text{ for } i = 0, \dots, N.$$

Note that this algebra is contained in the Borel sets of K . Also, since it is finitely generated, $\mathcal{C}$ is actually finite. And as a finite algebra, every set can be written as the union of the atoms inside that set. Denote by $\{a_1, \dots, a_l\} \subseteq B(K)$ the characteristic functions of the atoms in $\mathcal{C}$.

This observation implies that

$$\chi_{\{c_i \leq (f_s)_\sigma < c_{i+1}\}} = \sum_{j \in J(s, \sigma, i)} a_j,$$

for certain $J(s, \sigma, i) \subseteq \{1, \dots, l\}$, and similarly one defines $J(w, i)$ so that

$$\chi_{\{c_i \leq w < c_{i+1}\}} = \sum_{j \in J(w, i)} a_j.$$

Since $0 \leq (f_s)_\sigma, w \leq \mathbb{1}_K$, we have

$$(f_s)_\sigma = \sum_{i=0}^N (f_s)_\sigma \sum_{j \in J(s, \sigma, i)} a_j, \quad w = \sum_{i=0}^N w \sum_{j \in J(w, i)} a_j.$$

Let $A = \text{span}\{a_1, \dots, a_l\}$. We are going to define elements of A that approximate $(f_s)_\sigma$ and w . For every $s \in \{1, \dots, n\}$, $\sigma \in \{+, -\}$, and $j \in \{1, \dots, l\}$, define $c_{i(s, \sigma, j)} \in \{c_0, \dots, c_N\}$ to be the unique element of the set that satisfies

$$a_j \leq \chi_{\{c_{i(s, \sigma, j)} \leq (f_s)_\sigma < c_{i(s, \sigma, j)+1}\}}.$$

To make the notation shorter we will write $c_{i(s, \sigma, j)}$ as $c_{i(j)}^{s, \sigma}$ most of the time. Define

$$(f_s)_{\sigma, d} = \sum_{j=1}^l c_{i(j)}^{s, \sigma} a_j.$$

In other words, $(f_s)_{\sigma, d}$ is a discrete version of $(f_s)_\sigma$ that is constant on the sets corresponding to the characteristic functions $a_1, \dots, a_l$. More precisely, $(f_s)_{\sigma, d}$ is identically $c_{i(s, \sigma, j)}$ on the support of a_j , where $c_{i(s, \sigma, j)} \leq (f_s)_\sigma < c_{i(s, \sigma, j)+1}$ holds. In particular, $0 \leq (f_s)_{\sigma, d} \leq (f_s)_\sigma$.

Similarly, for $j \in \{1, \dots, l\}$ define $c_{t(j)} \in \{c_1, \dots, c_N\}$ to be the unique element of the set that satisfies

$$a_j \leq \chi_{\{c_{t(j)} \leq w < c_{t(j)+1}\}}$$

unless $a_j \leq \chi_{\{0 \leq w < c_1\}}$, in which case $c_{t(j)} = c_1$. This way we prevent the discrete version of w from vanishing at any point. With this change it is still true that $|c_{t(j)} - w| < \delta$ holds in the support of a_j for all $j = 1, \dots, l$. Define

$$w_d = \sum_{j=1}^l c_{t(j)} a_j.$$

Since the a_i are disjoint, A is a sublattice of $B(K)$. Define a product $\circ$ on A as follows: $a_i \circ a_j = 0$ if $i \neq j$, and $a_j \circ a_j = c_{t(j)} a_j$. It is immediate that this extends by linearity to a semiprime f -algebra product on A .

Fix $\varepsilon' > 0$. We are going to show that, for $\delta < \varepsilon'$, the product in A satisfies $|x \circ y| \leq |x \star y| + \varepsilon' \mathbb{1}_K$ for all $x, y \in A$ with $\|x\|_\infty, \|y\|_\infty \leq 1$. Let $x = \sum_{j=1}^l \lambda_j a_j$ and $y = \sum_{j=1}^l \mu_j a_j$ for some $\lambda_1, \dots, \lambda_l, \mu_1, \dots, \mu_l \in [-1, 1]$. Then

$$\begin{aligned} |x \circ y| &= \left| \sum_{j=1}^l \lambda_j \mu_j c_{t(j)} a_j \right| \\ &\leq \left| \sum_{j=1}^l \lambda_j \mu_j w a_j \right| + \sum_{j=1}^l |\lambda_j| |\mu_j| |w a_j - c_{t(j)} a_j| \\ &\leq |x \star y| + \mathbb{1}_K \left(\sup_{j=1, \dots, l} \sup_{\text{supp}(a_j)} |w - c_{t(j)}| \right) \\ &\leq |x \star y| + \delta \mathbb{1}_K. \end{aligned}$$

Let $k_1, \dots, k_n \geq 0$ be such that $k = k_1 + \dots + k_n \geq 1$. By definition:

$$\begin{aligned} (f_1)_{\sigma_1}^{k_1} \star \dots \star (f_n)_{\sigma_n}^{k_n} &= \sum_{j=1}^l (f_1)_{\sigma_1}^{k_1} \dots (f_n)_{\sigma_n}^{k_n} w^{k-1} a_j \\ (f_1)_{\sigma_1, d}^{k_1} \circ \dots \circ (f_n)_{\sigma_n, d}^{k_n} &= \sum_{j=1}^l (c_{i(j)}^{1, \sigma_1})^{k_1} \dots (c_{i(j)}^{n, \sigma_n})^{k_n} a_j \circ \dots \circ a_j \\ &= \sum_{j=1}^l (c_{i(j)}^{1, \sigma_1})^{k_1} \dots (c_{i(j)}^{n, \sigma_n})^{k_n} (c_{t(j)})^{k-1} a_j \end{aligned}$$

so that

$$\begin{aligned} \|(f_1)_{\sigma_1}^{k_1} \star \dots \star (f_n)_{\sigma_n}^{k_n} - (f_1)_{\sigma_1, d}^{k_1} \circ \dots \circ (f_n)_{\sigma_n, d}^{k_n}\|_\infty &= \\ \sup_{j=1, \dots, l} \sup_{\text{supp } a_j} |(f_1)_{\sigma_1}^{k_1} \dots (f_n)_{\sigma_n}^{k_n} w^{k-1} - (c_{i(j)}^{1, \sigma_1})^{k_1} \dots (c_{i(j)}^{n, \sigma_n})^{k_n} (c_{t(j)})^{k-1}|. \end{aligned}$$

But in $\text{supp } a_j$, $|(f_s)_{\sigma_s} - c_{i(j)}^{s, \sigma_s}| < \delta$ and $|w - c_{t(j)}| < \delta$. Using the fact that the function

$$\begin{aligned} [0, 1]^{n+1} &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n, y) &\longmapsto x_1^{k_1} \cdots x_n^{k_n} y^{k-1} \end{aligned}$$

is uniformly continuous, we can make the distance

$$\|(f_1)_{\sigma_1}^{k_1} \star \cdots \star (f_n)_{\sigma_n}^{k_n} - (f_1)_{\sigma_1, d}^{k_1} \circ \cdots \circ (f_n)_{\sigma_n, d}^{k_n}\|_\infty \quad (8.4)$$

arbitrarily small by choosing an appropriate $\delta > 0$.

According to Proposition 8.9, there exists an LL expression Ψ and $c \in \mathbb{N}$ such that

$$\Phi(\lambda_1, \dots, \lambda_n) = \Psi((\lambda_1)_{\sigma_1}^{j_1} \cdots (\lambda_n)_{\sigma_n}^{j_n} : j_1 + \cdots + j_n \leq c, \sigma_i \in \{+, -\})$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. This identity will also hold when evaluating the previous expression in arbitrary Banach f -algebras (Theorem 8.2). In particular, in $B(K)$,

$$f = \Psi((f_1)_{\sigma_1}^{j_1} \star \cdots \star (f_n)_{\sigma_n}^{j_n} : j_1 + \cdots + j_n \leq c, \sigma_i \in \{+, -\})$$

and, in A ,

$$\Phi((f_1)_d, \dots, (f_n)_d) = \Psi((f_1)_{\sigma_1, d}^{j_1} \circ \cdots \circ (f_n)_{\sigma_n, d}^{j_n} : j_1 + \cdots + j_n \leq c, \sigma_i \in \{+, -\}),$$

where we have defined $(f_s)_d = (f_s)_{+, d} - (f_s)_{-, d}$. Note that, since $(f_s)_{+, d} \wedge (f_s)_{-, d} \leq (f_s)_+ \wedge (f_s)_- = 0$, $(f_s)_{+, d}$ and $(f_s)_{-, d}$ are the positive and negative parts of $(f_s)_d$, respectively. In particular,

$$|(f_s)_d| = (f_s)_{+, d} + (f_s)_{-, d} \leq (f_s)_+ + (f_s)_- = |f_s|.$$

From the norm continuity of lattice-linear function calculus, and the fact that the quantities (8.4) can be made arbitrarily small for all $k_1 + \cdots + k_n \leq c$ by choosing $\delta > 0$ appropriately (keep in mind there are only finitely many non-negative integers $k_1, \dots, k_n$ satisfying this condition), it follows that

$$\|f - \Phi((f_1)_d, \dots, (f_n)_d)\|_\infty < \varepsilon. \quad \square$$

Proof of Theorem 8.69. Let $\{e_1, \dots, e_n\}$ be the canonical basis of ℓ_1^n . According to Theorem 8.56, $\text{FBfA}[\ell_1^n]$ can be isomorphically identified with $C(K)$ equipped with an appropriate product $\star$, where $K = [0, 1] \times S_\infty^n$. With this identification, the free norm $\|\cdot\|$ is equivalent to the uniform norm $\|\cdot\|_\infty$. Since $\text{FNfA}[\ell_1^n]$ is dense in $\text{FBfA}[\ell_1^n]$, it suffices to show the result for $g \in \text{FBfA}[\ell_1^n]$ living in $\text{FNfA}[\ell_1^n]$, the sublattice-algebra generated by $\eta_1 = \eta(e_1), \dots, \eta_n = \eta(e_n)$. Indeed, suppose the result holds in $\text{FNfA}[\ell_1^n]$, and let $g \in \text{FBfA}[\ell_1^n]$. Let $\varepsilon > 0$ and let $f \in \text{FNfA}[\ell_1^n]$ be such that $\|g - f\| < \varepsilon/3$. Let $T: \ell_1^n \rightarrow A$ be a contractive operator, where A is a semiprime finite-dimensional Banach f -algebra, such that $\|f\| \leq \|\hat{T}f\| + \varepsilon/3$. Then

$$\|\hat{T}f\| \leq \|\hat{T}(f - g)\| + \|\hat{T}g\| \leq \varepsilon/3 + \|\hat{T}g\|$$

and therefore

$$\|\hat{T}g\| \leq \|g\| \leq \|f\| + \varepsilon/3 \leq \|\hat{T}f\| + 2/3\varepsilon \leq \varepsilon + \|\hat{T}g\|.$$

Since $\varepsilon > 0$ is arbitrary, the result also follows for g .

So assume $f \in \text{FNfA}[\ell_1^n]$. There exists an LLA expression Φ such that $f = \Phi(\eta_1, \dots, \eta_n)$. Fix $\varepsilon > 0$. Then there exists $\varepsilon' > 0$ small enough such that, if $f_i = \eta_i/(1 + \varepsilon')$, then

$$\|f - \Phi(f_1, \dots, f_n)\| < \varepsilon.$$

Without loss of generality, we can assume $\varepsilon' < \varepsilon$; this will be useful later. Put $f' = \Phi(f_1, \dots, f_n)$ and apply the previous lemma to f' : there exist a finite-dimensional sublattice A of $B(K)$, together with a semiprime f -algebra product $\circ$ in A , and $(f_1)_d, \dots, (f_n)_d \in A$ satisfying $|(f_i)_d| \leq |f_i|$ such that

$$\|f' - (f')_d\|_\infty < \varepsilon \quad \text{where } (f')_d = \Phi((f_1)_d, \dots, (f_n)_d)$$

(the latter expression being evaluated inside A). Moreover, we can choose $\circ$ so that $|x \circ y| \leq |x \star y| + (\varepsilon'/\|\mathbf{1}_K\|)\mathbf{1}_K$ holds for all $x, y \in A$ with $\|x\|_\infty, \|y\|_\infty \leq 1$.

Note that the biduals of $C(K)$ with respect to either the free or the supremum norm coincide up to equivalence of norms, with the same equivalence constants. We shall denote the norms in the bidual the same way as the original ones. Therefore, making $\varepsilon > 0$ small as necessary, we can assume that $\|f' - (f')_d\| < \varepsilon$.

The sublattice A is a Banach lattice with respect to both the free and supremum norms. For every $x, y \in A$ with $\|x\|, \|y\| \leq 1$, in particular we have $\|x\|_\infty, \|y\|_\infty \leq 1$, and therefore the inequality $|x \circ y| \leq |x \star y| + (\varepsilon'/\|\mathbf{1}_K\|)\mathbf{1}_K$ holds. Taking free norms in this inequality:

$$\|x \circ y\| \leq \|x \star y\| + \varepsilon' \leq 1 + \varepsilon',$$

where in the second inequality we are using that the free norm is submultiplicative. Hence $\|x \circ y\| \leq (1 + \varepsilon')\|x\|\|y\|$ holds for general $x, y \in A$. Define on A the norm $\|\cdot\| = (1 + \varepsilon')\|\cdot\|$. This is certainly a lattice norm, and it is submultiplicative by the previous inequality. Hence A is a semiprime Banach f -algebra with respect to this norm.

Define a linear map $T: \ell_1^n \rightarrow A$ by $Te_i = (f_i)_d$ for $i = 1, \dots, n$. Since

$$\|(f_i)_d\| = (1 + \varepsilon')\|(f_i)_d\| \leq (1 + \varepsilon')\|f_i\| \leq 1$$

this map is contractive. Therefore it extends to a unique contractive lattice-algebra homomorphism $\hat{T}: \text{FBfA}[\ell_1^n] \rightarrow A$ satisfying

$$\hat{T}f = \Phi((f_1)_d, \dots, (f_n)_d) = (f')_d.$$

Putting all the estimations together:

$$\begin{aligned} \|\hat{T}f\| &= (1 + \varepsilon')\|(f')_d\| \\ &\leq (1 + \varepsilon')(\varepsilon + \|f'\|) \\ &\leq (1 + \varepsilon)(2\varepsilon + \|f\|). \end{aligned}$$

where in the last inequality we are also using that $\varepsilon' < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|\hat{T}f\|$ can be made arbitrarily close to $\|f\|$ by choosing appropriate A and T . This shows that $\tau(f) = \|f\|$. $\square$

Next we want to go one step further and show that τ coincides with the free norm in $\text{FBfA}[L_1(\mu)]$ for an arbitrary measure μ .

When $T: E \rightarrow F$ is a contractive operator between Banach spaces E and F , denote by $\bar{T}: \text{FBfA}[E] \rightarrow \text{FBfA}[F]$ the unique lattice-algebra homomorphism with $\|\bar{T}\| = \|T\|$ making the diagram

$$\begin{array}{ccc} \text{FBfA}[E] & \xrightarrow{\bar{T}} & \text{FBfA}[F] \\ \eta_E \uparrow & & \uparrow \eta_F \\ E & \xrightarrow{T} & F \end{array}$$

commute. Note that $\bar{T} = \widehat{\eta_F T}$. The following is a standard result relating T and $\bar{T}$.

Proposition 8.71. *Let E be a Banach space, let F be a subspace of E with a contractive projection map $P: E \rightarrow F$, and let $\iota: F \rightarrow E$ be the inclusion map. Then $\bar{P}: \text{FBfA}[E] \rightarrow \bar{\iota}(\text{FBfA}[F])$ is a contractive projection. In particular, $\bar{\iota}: \text{FBfA}[F] \rightarrow \text{FBfA}[E]$ is an isometric embedding.*

Proof. Since $P\iota = \text{id}_F$, uniqueness implies $\bar{P}\bar{\iota} = \text{id}_{\text{FBfA}[F]}$, so that $\bar{\iota}$ is injective and $\bar{P}$ is onto $\bar{\iota}(\text{FBfA}[F])$. From $P(\iota P) = P$ it follows that $\bar{P}$ is a projection, and since $\|\bar{\iota}\|, \|\bar{P}\| \leq 1$, $\bar{\iota}$ is an isometric embedding. $\square$

Proposition 8.72. *Let E be a Banach space. Suppose there exists a net $\{E_\lambda\}_\lambda$ of subspaces of E such that:*

- (i) *the closure of $\bigcup_\lambda E_\lambda$ is E ,*
- (ii) *there exist contractive projections $P_\lambda: E \rightarrow E_\lambda$,*
- (iii) *the free norm in $\text{FBfA}[E_\lambda]$ coincides with τ_{E_λ} .*

Then τ_E coincides with the free norm in $\text{FBfA}[E]$.

Proof. The result will follow once we show that $\tau(f) \geq \|f\|$ for every $f \in \text{FBfA}[E]$. Fix $\varepsilon > 0$ and let $g \in \text{FNfA}[E]$ be such that $\|f - g\| < \varepsilon/2$. There exists an LLA expression Φ and $x_1, \dots, x_n \in E$ satisfying $g = \Phi(\eta_{x_1}, \dots, \eta_{x_n})$. Choose $z_1, \dots, z_n \in \bigcup_\lambda E_\lambda$ with $\|z_i - x_i\| < \delta$, this $\delta > 0$ to be determined later. Since $\{E_\lambda\}_\lambda$ is a net, there exists some $G = E_{\lambda_0}$ for which $z_1, \dots, z_n \in G$. Let $h = \Phi(\eta_{z_1}, \dots, \eta_{z_n})$. Since the operations in $\text{FBfA}[E]$ are norm continuous, $\|g - h\| < \varepsilon/2$ as long as we choose $\delta > 0$ small enough. Hence $\|f - h\| < \varepsilon$.

According to Proposition 8.71, one can view $\text{FBfA}[G]$ as a closed sublattice-algebra of $\text{FBfA}[E]$; with this identification, $h \in \text{FBfA}[G]$, and $\|h\|_{\text{FBfA}[E]} = \|h\|_{\text{FBfA}[G]}$. By assumption, there exists a semiprime finite-dimensional Banach

f -algebra A , and a contractive operator $T: G \rightarrow A$, such that $\|\hat{T}h\|_A \geq \|h\| - \varepsilon$. Then $TP_{\lambda_0}: E \rightarrow A$ is a contractive operator such that

$$\widehat{TP_{\lambda_0}}h = \Phi(TP_{\lambda_0}z_1, \dots, TP_{\lambda_0}z_n) = \Phi(Tz_1, \dots, Tz_n) = \hat{T}h$$

and

$$\|\hat{T}h\|_A = \|\widehat{TP_{\lambda_0}}h\|_A \leq \|f - h\| + \|\widehat{TP_{\lambda_0}}f\|_A < \varepsilon + \|\widehat{TP_{\lambda_0}}f\|_A$$

so that

$$\tau(f) \geq \|\widehat{TP_{\lambda_0}}f\|_A > \|\hat{T}h\|_A - \varepsilon \geq \|h\| - 2\varepsilon \geq \|f\|_{\text{FBfA}[E]} - 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows that $\tau(f) = \|f\|$. $\square$

Remark 8.73. Suppose that F is a contractively complemented subspace of E and that τ_E coincides with the free norm in $\text{FBfA}[E]$. We are going to show that τ_F also coincides with the free norm in $\text{FBfA}[F]$. Let $P: E \rightarrow F$ be a contractive projection. By Proposition 8.71, the norms $\|\cdot\|_{\text{FBfA}[F]}$ and $\|\cdot\|_{\text{FBfA}[E]}$ coincide on $\text{FBfA}[F]$ when the latter is viewed as a sublattice-algebra of $\text{FBfA}[E]$. So it suffices to check that $\tau_F(f) \geq \tau_E(f)$ for every $f \in \text{FBfA}[F]$. Since τ is continuous with respect to the free norm (this is easy to see using that τ is a seminorm and that $\tau \leq \|\cdot\|$), it suffices to check the inequality for $f \in \text{FNfA}[F]$. Let A be a semiprime finite-dimensional Banach f -algebra, and let $T: E \rightarrow A$ be a contractive operator. Its restriction $S: F \rightarrow A$ is also a contractive operator such that $\hat{S}f = \hat{T}f$ for all $f \in \text{FNfA}[F]$. Therefore $\|\hat{T}f\| = \|\hat{S}f\| \leq \tau_F(f)$. Since T and A were arbitrary, $\tau_E(f) \leq \tau_F(f)$.

Corollary 8.74. *The norm in $\text{FBfA}[L_1(\mu)]$ is τ for every measure μ .*

Proof. In $L_1(\mu)$ there exists a net of subspaces, each isometric to ℓ_1^n for some $n \in \mathbb{N}$, satisfying the conditions in Proposition 8.72. The result then follows from Theorem 8.69. $\square$

8.4 REPRESENTATION OF FREE BANACH f -ALGEBRAS IN SPACES OF CONTINUOUS FUNCTIONS

Recall from Corollary 8.51 that, for every Banach space E , we have an injective and contractive lattice-algebra homomorphism $\iota: \text{FNfA}[E] \rightarrow C(B_{E^*})$. This map can be extended to the completion $\hat{\iota}: \text{FBfA}[E] \rightarrow C(B_{E^*})$. The extension is certainly contractive and a lattice-algebra homomorphism. But is it injective? When it is, we will say that $\text{FBfA}[E]$ is *representable in $C(B_{E^*})$* . Since ι is never an embedding (Remark 8.61), it is not clear at all whether $\text{FBfA}[E]$ is representable in $C(B_{E^*})$ or not.

Compare this with the situation for the free vector lattice $\text{FVL}[E]$, whose restriction map $\text{FVL}[E] \rightarrow C(B_{E^*})$ extends to an injective and contractive lattice-algebra homomorphism $\text{FBL}[E] \rightarrow C(B_{E^*})$ (see Section 1.6). Once $\text{FBL}[E]$ has been represented inside $C(B_{E^*})$, many of its properties follow, as they depend only on this representation rather than on the explicit expression of the norm

(see [OTTT24]). Therefore, even though no explicit expression for the norm in $\text{FBfA}[E]$ is available, we will be able to recover several properties analogous to those of $\text{FBL}[E]$ as long as we assume that $\text{FBfA}[E]$ is representable in $C(B_{E^*})$ (this will be done in Section 8.5).

In this section we explore when $\text{FBfA}[E]$ is representable. The space $\text{FBfA}[E]$ is quite difficult to understand, as it is the completion of $\text{FNfA}[E]$ with respect to a certain abstract norm. First, we characterize those E for which $\text{FBfA}[E]$ is representable in $C(B_{E^*})$ as those for which $\text{FBfA}[E]$ is semiprime. Representability is thus reduced to a purely algebraic question. However, whether or not $\text{FBfA}[E]$ is semiprime is still not obvious, since the completion of a semiprime normed f -algebra need not be semiprime. At this point, the technical computations involving the seminorm τ carried out in the previous section come to rescue: we show that, if τ defines a norm on $\text{FBfA}[E]$, then $\text{FBfA}[E]$ is semiprime.

Theorem 8.75. *Let E be a Banach space. The map $\iota: \text{FNfA}[E] \rightarrow C(B_{E^*})$ extends to an injective lattice-algebra homomorphism $\hat{\iota}: \text{FBfA}[E] \rightarrow C(B_{E^*})$ if and only if $\text{FBfA}[E]$ is semiprime.*

Proof. Suppose first that the map $\iota: \text{FNfA}[E] \rightarrow C(B_{E^*})$ extends to an injective lattice-algebra homomorphism $\hat{\iota}: \text{FBfA}[E] \rightarrow C(B_{E^*})$. Let $f \in N(\text{FBfA}[E])$. For every $x^* \in B_{E^*}$, $x^* \neq 0$, there exists some $x \in B_E$ for which $\iota(\eta_x)(x^*) = x^*(x) \neq 0$. Since $f\eta_x = 0$, it follows that

$$0 = \hat{\iota}(f\eta_x)(x^*) = \hat{\iota}(f)(x^*)\iota(\eta_x)(x^*).$$

Thus $\hat{\iota}(f)(x^*) = 0$ for every $x^* \in B_{E^*} \setminus \{0\}$, and by continuity $\hat{\iota}(f)(0) = 0$. So $\hat{\iota}(f) = 0$ and, since we are assuming $\hat{\iota}$ to be injective, $f = 0$.

Conversely, suppose $\text{FBfA}[E]$ is semiprime. The left regular representation $L: \text{FBfA}[E] \rightarrow \mathcal{Z}(\text{FBfA}[E])$, which sends $f \in \text{FBfA}[E]$ to the left multiplication operator $L_f(g) = fg$, $g \in \text{FBfA}[E]$, is a contractive lattice-algebra homomorphism. Since $\text{FBfA}[E]$ is semiprime, L is also injective. Let Z be the closure of $\text{VLA}(\{L_f : f \in \text{FBfA}[E]\} \cup \{\text{id}_{\text{FBfA}[E]}\})$ in $\mathcal{Z}(\text{FBfA}[E])$ (that is, the closure with respect to the uniform norm that induces $\text{id}_{\text{FBfA}[E]}$, which coincides with the operator norm in $\mathcal{Z}(\text{FBfA}[E])$; see Theorem 1.8). Being a Banach f -algebra with identity and having the uniform norm with respect to this identity, Z is lattice-algebra isometric to $C(K)$ for a certain compact Hausdorff space K (Theorem 3.1). Restricting the range of L yields a contractive and injective lattice-algebra homomorphism $L: \text{FBfA}[E] \rightarrow C(K)$.

Along this proof, ϕ_t for $t \in K$ shall denote the point-mass measure at t . The composition map $\phi_t L: \text{FBfA}[E] \rightarrow \mathbb{R}$ is a contractive lattice-algebra homomorphism. Hence it is uniquely determined by $\phi_t L\eta: E \rightarrow \mathbb{R}$, which is an element of B_{E^*} . Define the map

$$\begin{aligned} \psi: \quad K &\longrightarrow (B_{E^*}, w^*) \\ t &\longmapsto \phi_t L\eta \end{aligned}.$$

Note that $\psi(t) = (L\eta)^* \phi_t$. If we identify K homeomorphically with $\{\phi_t : t \in K\}$, then the map ψ is nothing but the restriction of $(L\eta)^*$ to $\{\phi_t : t \in K\}$.

Since $L\eta$ is continuous, its adjoint $(L\eta)^*$ is weak* to weak* continuous. Hence ψ is continuous. It is also injective: if $\psi(t) = \psi(s)$, then for all $x \in E$, $\phi_t(L_{\eta_x}) = \phi_s(L_{\eta_x})$, and since the point-mass measures are lattice-algebra homomorphisms, ϕ_t and ϕ_s coincide on $L(\text{FNfA}[E])$. Since $L(\text{FNfA}[E])$ is dense in $L(\text{FBfA}[E])$, by continuity they must also coincide on $L(\text{FBfA}[E])$. Also, $\phi_t(\text{id}_{\text{FBfA}[E]}) = 1 = \phi_s(\text{id}_{\text{FBfA}[E]})$. Since $L(\text{FBfA}[E])$ and $\text{id}_{\text{FBfA}[E]}$ generate, as a Banach lattice algebra, the whole $C(K)$, it follows that $\phi_t = \phi_s$ and therefore that $t = s$.

Being K compact, ψ is actually an embedding. That is, K is homeomorphic to its image $\psi(K) \subseteq B_{E^*}$. Note that $(L\eta)(x) = \iota(\eta_x) \circ \psi$. In other words: the map

$$\begin{aligned} S: E &\longrightarrow C(\psi(K)) \\ x &\longmapsto \iota(\eta_x)|_{\psi(K)} \end{aligned}$$

extends to a lattice-algebra homomorphism $\hat{S}: \text{FBfA}[E] \rightarrow C(\psi(K))$ which is injective. Indeed, $C_{\psi^{-1}} \circ L$, where $C_{\psi^{-1}}: C(K) \rightarrow C(\psi(K))$ denotes the composition map, is an injective and contractive lattice-algebra homomorphism satisfying $(C_{\psi^{-1}} \circ L)\eta(x) = \iota(\eta_x)|_{\psi(K)}$. By uniqueness, $\hat{S}$ is precisely $C_{\psi^{-1}} \circ L$.

Consider the natural restriction map $R: C(B_{E^*}) \rightarrow C(\psi(K))$, which is also a contractive lattice-algebra homomorphism. The diagram

$$\begin{array}{ccc} \text{FBfA}[E] & \xrightarrow{\hat{S}} & C(\psi(K)) \\ & \searrow i & \nearrow R \\ & C(B_{E^*}) & \end{array}$$

commutes, because $\hat{S}(\eta_x) = \iota(\eta_x)|_{\psi(K)} = (Ri)(\eta_x)$ and all arrows are contractive lattice-algebra homomorphisms. Since $\hat{S}$ has been shown to be injective, so must be $\hat{\iota}$. $\square$

Corollary 8.76. *If E is a finite-dimensional Banach space, then $\text{FBfA}[E]$ is representable.*

Proof. This is a consequence of Corollary 8.62. $\square$

Even though $\text{FNfA}[E]$ is semiprime for every Banach space E , it is not clear at all whether its completion $\text{FBfA}[E]$ is also semiprime. In general, it is not true that the completion of a semiprime normed f -algebra is semiprime. The following lemma provides a general procedure to construct examples of such algebras.

Lemma 8.77. *Let B be a semiprime Banach f -algebra. Let $A \subseteq \mathbb{R}_0 \oplus_{\infty} B$ be a sublattice-algebra such that $(1, 0) \notin A$ but $(1, 0) \in \bar{A}$. Then A is a semiprime normed f -algebra whose completion is not semiprime.*

Proof. Since $(1, 0) \in \bar{A}$, and $(1, 0)^2 = (0, 0)$, $\bar{A}$ is not semiprime. To show that A is semiprime, let $(\lambda, a) \in A$ be such that $(\lambda, a)^2 = (0, 0)$. Then $a^2 = 0$ and, since B is semiprime, $a = 0$. Thus $(\lambda, a) = \lambda(1, 0) \in A$, but we are assuming $(1, 0) \notin A$, so that $\lambda = 0$. $\square$

Below we present two explicit examples of this construction. The first uses a limiting process to guarantee that $(1, 0) \notin A$, while the second uses the freeness of the generators.

Example 8.78. Let

$$A = \{ (\lambda, (x_n)) \in \mathbb{R}_0 \oplus c : \lim nx_n = \lambda \}.$$

It is clear that this is a sublattice of $\mathbb{R}_0 \oplus c$. It is also a subalgebra, for if $(\lambda, (x_n)), (\mu, (y_n)) \in A$, then their product is $(0, (x_n y_n))$ and, since $\lim x_n = 0$, $\lim x_n y_n = 0$. Clearly $(1, 0) \notin A$. For every $n \in \mathbb{N}$, the sequences

$$x^{(n)} = (0, \dots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \dots),$$

where the zeros are in the first n coordinates, are such that $x^{(n)} \rightarrow 0$ in c . Moreover, $(1, x^{(n)}) \in A$. Therefore $\lim(1, x^{(n)}) = (1, 0) \in \bar{A}$. By Lemma 8.77, A is a semiprime normed f -algebra whose completion is not semiprime.

Example 8.79. Consider the Banach f -algebra $B = \mathbb{R}_0 \oplus_\infty \text{FBfA}[\ell_1]$. Denote by e_n , $n \geq 1$, the canonical basis of ℓ_1 . Let

$$A = \text{VLA}\{(1, \eta_{e_n/n}) : n \in \mathbb{N}\} \subseteq B.$$

It is clear that A is a sublattice-algebra. Its completion contains $(1, 0)$, because $\|\eta_{e_n/n}\| = \|e_n/n\|_1 = 1/n$ tends to zero.

Let us show that $(1, 0) \notin A$. Suppose, in order to reach a contradiction, that $(1, 0) \in A$. Then there exists an LLA expression $\Phi[t_1, \dots, t_n]$ such that $(1, 0) = \Phi((1, \eta_{e_1}), \dots, (1, \eta_{e_n/n}))$. Since the operations are computed coordinatewise:

$$(1, 0) = (\Phi(1, \dots, 1), \Phi(\eta_{e_1}, \dots, \eta_{e_n/n})). \quad (8.5)$$

We are going to show that Φ vanishes on $[-1/n, 1/n]^n \subseteq \mathbb{R}^n$. Fix $(\lambda_1, \dots, \lambda_n) \in [-1/n, 1/n]^n$. The element

$$x^* = (\lambda_1, 2\lambda_2, \dots, n\lambda_n, 0, 0, \dots) \in B_{\ell_\infty}$$

seen as a contractive functional on ℓ_1 extends to a lattice-algebra homomorphism $\widehat{x^*}: \text{FBfA}[\ell_1] \rightarrow \mathbb{R}$. Therefore

$$0 = \widehat{x^*}(\Phi(\eta_{e_1}, \dots, \eta_{e_n/n})) = \Phi(x^*(e_1), \dots, x^*(e_n)/n) = \Phi(\lambda_1, \dots, \lambda_n).$$

This proves that Φ vanishes on $[-1/n, 1/n]^n$. By Lemma 8.7, Φ must also vanish on $\mathbb{R}_0$. This contradicts the fact that, according to (8.5), $1 = \Phi(1, \dots, 1)$ in $\mathbb{R}_0$. Hence $(1, 0) \notin A$. By Lemma 8.77, A is an example of a semiprime normed f -algebra whose completion is not semiprime.

The previous example illustrates why determining whether the completion of a semiprime normed f -algebra is again semiprime or not is far from trivial. The next proposition shows how τ can be very useful in that matter.

Proposition 8.80. *Let E be a Banach space. Suppose τ defines a norm on $\text{FBfA}[E]$. Then $\text{FBfA}[E]$ is semiprime.*

Proof. This proof is based on the observation that every semiprime finite-dimensional f -algebra A has an identity. Indeed, if $\{e_1, \dots, e_n\}$ is a basis of pairwise disjoint elements of A , then $e_i^2 = w_i e_i$ for some $w_i \geq 0$, and since A is semiprime, $w_i > 0$. The identity is the element $\sum_{i=1}^n e_i/w_i$.

Let $f \in N(\text{FBfA}[E])$. Let A be a semiprime finite-dimensional Banach f -algebra and let $T: E \rightarrow A$ be a contractive operator. Since $\text{range } \hat{T}$ is a sublattice-algebra of A , in particular is semiprime, and by the initial observation has an identity. Let g be in the preimage of the identity. Then

$$\hat{T}f = \hat{T}f\hat{T}g = \hat{T}(fg) = 0.$$

Since T and A were arbitrary, $\tau(f) = 0$, and since we are assuming τ defines a norm on $\text{FBfA}[E]$, $f = 0$. $\square$

Corollary 8.81. *Let F be a contractively complemented subspace of a Banach space E . Suppose there exists a net $\{E_\lambda\}_\lambda$ of subspaces of E such that:*

- (i) *the closure of $\bigcup_\lambda E_\lambda$ is E ,*
- (ii) *there exist contractive projections $P_\lambda: E \rightarrow E_\lambda$,*
- (iii) *the free norm in $\text{FBfA}[E_\lambda]$ coincides with τ_{E_λ} .*

Then $\text{FBfA}[F]$ is representable in $C(B_{F^*})$.

Proof. By Proposition 8.72, τ is a norm on the Banach f -algebras generated by these Banach spaces. The result follows from a successive application of Proposition 8.80 and Theorem 8.75. $\square$

Corollary 8.82. *$\text{FBfA}[L_1(\mu)]$ is representable in $C(B_{L_\infty(\mu)})$ for every measure μ .*

Remark 8.83. In particular, everything we have done so far applies to $\text{FBfA}(S) = \text{FBfA}[\ell_1(S)]$ for every set S . Hence, from the point of view of the free Banach f -algebra generated by a set, we are being completely general.

After this discussion, we have to leave the following open.

Question 8.84. Is $\text{FBfA}[E]$ semiprime for every Banach space E ?

8.5 PROPERTIES OF THE FREE BANACH f -ALGEBRA GENERATED BY A BANACH SPACE

In this section we investigate the lattice and algebraic properties of the free Banach f -algebra generated by a Banach space.

8.5.1 Basic properties

Proposition 8.85. *Let E be a Banach space. Neither $\text{FNfA}[E]$ nor $\text{FBfA}[E]$ have an approximate identity.*

Proof. Suppose (h_α) were an approximate identity in $\text{FBfA}[E]$. Let $x^* \in B_{E^*}$ be a non-zero functional. Consider it as an operator $x^*: E \rightarrow \mathbb{R}_0$, and extend it to a contractive lattice-algebra homomorphism

$$\widehat{x^*}: \text{FBfA}[E] \rightarrow \mathbb{R}_0.$$

Let $x \in E$ be such that $x^*(x) \neq 0$. Since $\eta_x h_\alpha \rightarrow \eta_x$ in $\text{FBfA}[E]$, it follows that

$$\widehat{x^*}(\eta_x) \widehat{x^*}(h_\alpha) \rightarrow \widehat{x^*}(\eta_x) = x^*(x) \neq 0.$$

This is impossible because in $\mathbb{R}_0$ the product of any two elements is 0. The same argument works for $\text{FNfA}[E]$. $\square$

The following two properties are analogous to Proposition 1.36. In this context, however, one has to be careful with the fact that we do not know whether $\text{FBfA}[E]$ is representable in $C(B_{E^*})$ or not.

Proposition 8.86. *Let E be a Banach space.*

- (i) *E is finite-dimensional if and only if $\text{FBfA}[E]$ has a strong unit.*
- (ii) *E is separable if and only if $\text{FBfA}[E]$ has a quasi-interior point.*

Proof. (i) When E is finite-dimensional, it was shown in Theorem 8.56 that $\text{FBfA}[E]$ has a strong unit. Conversely, suppose $e \in \text{FBfA}[E]$ is a strong unit but $\dim E = \infty$. Scaling e , we may assume that $|\eta_x| \leq e$ for all $x \in B_E$. Let $f \in \text{FNfA}[E]$ be such that $\|e - f\| < 1/2$. Then $\|\widehat{ie} - \widehat{if}\|_\infty < 1/2$. Since $f \in \text{FNfA}[E]$, $\widehat{if} = \Phi(\iota(\eta_{x_1}), \dots, \iota(\eta_{x_n}))$ for some LLA expression Φ and some $x_1, \dots, x_n \in E$. We are assuming E to be infinite-dimensional, so there exists $x^* \in S_{E^*}$ that vanishes on $x_1, \dots, x_n$. Hence $(\widehat{if})(x^*) = 0$. For every $x \in B_E$:

$$\begin{aligned} |x^*(x)| &= \iota(|\eta_x|)(x^*) \\ &\leq \widehat{ie}(x^*) \\ &= |(\widehat{ie})(x^*) - (\widehat{if})(x^*)| \\ &\leq \|\widehat{ie} - \widehat{if}\|_\infty < 1/2. \end{aligned}$$

This contradicts the fact that $\|x^*\| = 1$.

- (ii) Suppose E is separable. Then $\eta(E)$ is separable and, since $\text{FBfA}[E]$ is the closed sublattice-algebra generated by $\eta(E)$, it is also separable. In particular, it has a quasi-interior point.

Conversely, suppose $e \in \text{FBfA}[E]$ is a quasi-interior point. If $x^* \in B_{E^*}$ is such that $\widehat{ie}(x^*) = 0$, then $\widehat{if}(x^*) = 0$ for all $f \in I_e$. Since I_e is dense in

$\text{FBfA}[E]$, it follows that $\hat{t}f(x^*) = 0$ for all $f \in \text{FBfA}[E]$. In particular, $x^*(x) = \hat{t}\eta_x(x^*) = 0$ for all $x \in E$. This shows that $\hat{t}e$ only vanishes at 0.

For every $n \in \mathbb{N}$, let $U_n = \{x^* \in B_{E^*} : \hat{t}e(x^*) < 1/n\}$. Then U_n is a weak* open subset of B_{E^*} and $\bigcap_{n=1}^{\infty} U_n = \{0\}$. For every n , let $A_n \subseteq E$ be a finite set such that U_n contains

$$V_n = \{x^* \in B_{E^*} : |x^*(x)| < 1 \text{ for all } x \in A_n\}.$$

Let F be the closed span of $\bigcup_{n=1}^{\infty} A_n$. We claim that $F = E$. Indeed, if F were not E , then there would exist some $x^* \in S_{E^*}$ vanishing on F . On the other hand, since $\bigcap_{n=1}^{\infty} V_n = \{0\}$, there exists some n for which $x^* \notin V_n$, that is, $|x^*(x)| \geq 1$ for some $x \in A_n$. Since $A_n \subseteq F$, this is a contradiction. $\square$

8.5.2 Order density

In the previous section, we insisted that the representability of $\text{FBfA}[E]$ in $C(B_{E^*})$ (equivalently, the semiprimeness of $\text{FBfA}[E]$) is key in showing many properties analogous to those studied for the free Banach lattice in Section 1.6. The upcoming results will show clearly why this is the case. The next proposition, which is the analogue of Proposition 1.37, follows from a topological argument identical to the proof of Proposition 8.12 once we use the representation $\hat{t}: \text{FBfA}[E] \rightarrow C(B_{E^*})$.

Proposition 8.87. *Let E be a Banach space for which $\text{FBfA}[E]$ is semiprime.*

- (i) *For every $x \in E$, $x \neq 0$, $|\eta_x|$ is a weak order unit in $\text{FBfA}[E]$.*
- (ii) *If E has dimension strictly greater than one, then $\text{FBfA}[E]$ has no non-trivial projection bands.*
- (iii) *If E has dimension strictly greater than one, then $\text{FBfA}[E]$ is not σ -order complete and has no atoms.*

Even though the free vector lattice is not order dense in the free Banach lattice unless the base space is finite-dimensional [AD25], the free normed f -algebra is always order dense in the free Banach f -algebra whenever the latter is representable. Showing this requires first the following technical fact.

Lemma 8.88. *Let E be a Banach space and let $U \subseteq B_{E^*}$ be a weak* open set. Then there exists an element $f \in \iota(\text{FNfA}[E])$, $f > 0$, such that $\text{supp } f \subseteq U$ and $\|f\|_{\infty} \leq 1$.*

Proof. For every $a < b$, the support of the real function $x \mapsto [x_+(x_+ - a)(x_+ - b)]_-$ is contained in $[a, b] \cap [0, \infty)$. Define, depending on the interval (a, b) , the LLA expression

$$B_{(a,b)}[t] = \begin{cases} [t_+(t_+ - a)(t_+ - b)]_- & \text{if } b > 0, \\ [t_-(t_- + a)(t_- + b)]_- & \text{if } b \leq 0. \end{cases}$$

Note that $x \mapsto B_{(a,b)}(x)$ defines a non-zero real function whose support is $[a, b] \cap [0, \infty)$ if $b > 0$ and $[a, b]$ otherwise. Normalize it so as to have $0 \leq B_{(a,b)}(x) \leq 1$ for all $x \in \mathbb{R}$.

Making U smaller, if necessary, we may assume that there exist $x_1, \dots, x_n \in B_E$ linearly independent, $\delta > 0$ and $x^* \in E^*$ with $\|x^*\| < 1$ such that

$$U = \{y^* \in B_{E^*} : |y^*(x_i) - x^*(x_i)| < \delta \text{ for } i = 1, \dots, n\}.$$

For a fixed $0 < \delta' \leq \delta$ to be determined later, define

$$f = \bigwedge_{i=1}^n B_{(x^*(x_i) - \delta'/2, x^*(x_i) + \delta'/2)}(\eta_{x_i}).$$

The most difficult part is to show that $f \neq 0$. For this, define a functional y^* on $F = \text{span}\{x_1, \dots, x_n\}$ in the following way. If $|x^*(x_i)| \geq \delta'/2$, then set $\varepsilon_i = 0$ and $y^*(x_i) = x^*(x_i) + \varepsilon_i$. If instead $-\delta'/2 < x^*(x_i) < \delta'/2$, then there exists $0 \leq \varepsilon_i < \delta'/2$ such that $x^*(x_i) + \varepsilon_i > 0$; in this case, set $y^*(x_i) = x^*(x_i) + \varepsilon_i$.

We are going to show that, for δ' small enough, $\|y^*\| \leq 1$. Let $C > 0$ be such that $\|\cdot\|_\infty \leq C\|\cdot\|$ in F , where $\|\sum_{i=1}^n \lambda_i x_i\|_\infty = \max\{|\lambda_i| : 1 \leq i \leq n\}$. For $x \in B_F$, $x = \sum_{i=1}^n \lambda_i x_i$, compute

$$\begin{aligned} |y^*(x)| &= \left| \sum_{i=1}^n \lambda_i (x^*(x_i) + \varepsilon_i) \right| \\ &= \left| x^*(x) + \sum_{i=1}^n \lambda_i \varepsilon_i \right| \\ &\leq \|x^*\| \|x\| + \|x\|_\infty \sum_{i=1}^n \varepsilon_i \\ &\leq \|x\| (\|x^*\| + Cn\delta'/2). \end{aligned}$$

Recall that $\|x^*\| < 1$ was fixed at the beginning, and that n and C are determined by $x_1, \dots, x_n$, which were also chosen before δ' . Thus we may set δ' small enough so as to have $\|y^*\| \leq 1$. Use Hahn–Banach to extend it to a functional $y^* \in B_{E^*}$. This functional satisfies

$$B_{(x^*(x_i) - \delta'/2, x^*(x_i) + \delta'/2)}(y^*(x_i)) > 0$$

for every $i = 1, \dots, n$, and so $f(y^*) > 0$. This shows that $f \neq 0$. Now we check the remaining properties of f . By construction, for every $z^* \in B_{E^*}$, $f(z^*) \geq 0$, and $f(z^*) \neq 0$ implies $z^*(x_i) \in (x^*(x_i) - \delta/2, x^*(x_i) + \delta/2)$ for $i = 1, \dots, n$. Therefore $f \geq 0$ and $\text{supp } f \subseteq U$. $\square$

Proposition 8.89. *Let E be a Banach space. Then $\iota(\text{FNfA}[E])$ is order dense in $C(B_{E^*})$.*

Proof. Let $f \in C(B_{E^*})$, $f > 0$, and let $x^* \in B_{E^*}$ be such that $f(x^*) \neq 0$. By continuity, there exists an $\varepsilon > 0$ and a weak* open neighbourhood U of x^* such that $f(y^*) > \varepsilon$ whenever $y^* \in U$. Let $g \in \iota(\text{FNfA}[E])$ be an element satisfying the conditions of Lemma 8.88 for the open set U . Then $0 < \varepsilon g \leq f$. $\square$

Corollary 8.90. *Let E be a Banach space for which $\text{FBfA}[E]$ is semiprime. Then $\text{FNfA}[E]$ is order dense in $\text{FBfA}[E]$.*

Proof. Let $f \in \text{FBfA}[E]$, $f > 0$. By Theorem 8.75, we have that $\hat{i}: \text{FBfA}[E] \rightarrow C(B_{E^*})$ is injective, so $\hat{i}(f) > 0$. By Proposition 8.89 there exists $g \in \text{FNfA}[E]$ such that $0 < \hat{i}(g) \leq \hat{i}(f)$. Again the injectivity of $\hat{i}$ implies $0 < g \leq f$. $\square$

Proposition 8.91. *For every Banach space E , $\text{FNfA}[E]$ has the countable chain condition. If $\text{FBfA}[E]$ is semiprime, then it also has the countable chain condition.*

Proof. That $\text{FNfA}[E]$ has the countable chain condition follows from the existence of the representation $\iota_E: \text{FNfA}[E] \rightarrow C(B_{E^*})$ and a topological argument as in Remark 8.14. If $\text{FBfA}[E]$ is semiprime, then $\text{FNfA}[E]$ is order dense in $\text{FBfA}[E]$, and therefore $\text{FBfA}[E]$ also has the countable chain condition. $\square$

8.5.3 Extension of operators

The following is an analogue of Lemma 1.38 with essentially the same proof.

Lemma 8.92. *Let E and F be Banach spaces for which $\text{FBfA}[E]$ and $\text{FBfA}[F]$ are semiprime. Let $T: E \rightarrow F$ be a contractive operator. Then the extension operator $\bar{T}: \text{FBfA}[E] \rightarrow \text{FBfA}[F]$ satisfies $\hat{i}_F(\bar{T}f) = \hat{i}_E(f) \circ T^*$.*

Proof. The composition operator

$$\begin{aligned} C_{T^*}: \text{FBfA}[E] &\longrightarrow C(B_{F^*}) \\ f &\longmapsto \hat{i}_E(f) \circ T^* \end{aligned}$$

is a lattice-algebra homomorphism. Since $C_{T^*}(\eta_x) = \iota_F(\eta_{Tx})$, C_{T^*} maps $\text{FNfA}[E]$ to $\iota_F(\text{FNfA}[F])$ contractively, so the range of C_{T^*} is contained in $\hat{i}_F(\text{FBfA}[F])$. By uniqueness, $C_{T^*} = \hat{i}_F \circ \bar{T}$. $\square$

Recall from Proposition 1.39 that an operator between Banach spaces $T: E \rightarrow F$ is injective, onto or has dense range if and only if the induced operator between the free Banach lattices $\bar{T}: \text{FBL}[E] \rightarrow \text{FBL}[F]$ is injective, onto or has dense range, respectively. The following example shows that for free Banach f -algebras the situation is different.

Example 8.93. Let $T: \ell_1^2 \rightarrow \ell_\infty^2$ be the formal identity. Identify both $B_{\ell_1^2}$ and $B_{\ell_\infty^2}$ as subsets of the Euclidean plane $\mathbb{R}^2$. Identify also $\text{FBfA}[\ell_1^2]$ and $\text{FBfA}[\ell_\infty^2]$ as sublattice-algebras of $C(B_{\ell_\infty^2})$ and $C(B_{\ell_1^2})$ through $\hat{i}_{\ell_1^2}$ and $\hat{i}_{\ell_\infty^2}$, respectively (Corollary 8.76 ensures that this is possible). The induced operator $\bar{T}: \text{FBfA}[\ell_1^2] \rightarrow \text{FBfA}[\ell_\infty^2]$ is, according to Lemma 8.92, nothing more than the restriction map $\bar{T}f = f|_{B_{\ell_1^2}}$. By Lemma 8.88, there exists a non-zero $f \in \text{FBfA}[\ell_1^2]$ such that $\text{supp } f \subseteq [1/2, 1]^2$. Then $\bar{T}f = 0$, so that $\bar{T}$ is not injective, even though T is.

Nevertheless, we have the following positive results.

Proposition 8.94. *Let E and F be Banach spaces and let $T: E \rightarrow F$ be a contractive operator.*

- (i) $\bar{T}$ has dense range if and only if T has dense range.
- (ii) If T is a quotient map, then so is $\bar{T}$.
- (iii) If $\bar{T}$ is a bijection, then so is T .
- (iv) With the additional assumption that $\text{FBfA}[E]$ and $\text{FBfA}[F]$ are semiprime, $\bar{T}$ is injective if and only if $T^*(B_{F^*})$ is w^* -dense in B_{E^*} .

Proof. The proofs of (i) and (ii) are very similar to those in [OTT24, Proposition 2.3]. It is the proofs of (iii) and (iv) that require a different approach.

- (i) Suppose first that T has dense range. For every $y \in F$ and $\varepsilon > 0$ there exists $x \in E$ such that $\|Tx - y\| < \varepsilon$. This is the same as $\|\bar{T}\eta_x - \eta_y\| < \varepsilon$, which implies that $\overline{\text{range } \bar{T}}$ contains $\{\eta_y : y \in F\}$. Since $\bar{T}$ is a lattice-algebra homomorphism, $\overline{\text{range } \bar{T}}$ is a closed sublattice-algebra. Hence it must be $\text{FBfA}[F]$.

To show the converse implication, suppose $\text{range } T$ is not dense, and let $y^* \in B_{F^*}$ be a non-zero functional that vanishes on it. Then the map $\hat{y}^* \in \text{FBfA}[F]^*$ vanishes on $\bar{T}\eta_x$ for every $x \in E$. Since $\hat{y}^*$ is a lattice-algebra homomorphism, it vanishes on $\overline{\text{range } \bar{T}}$. Hence the range of $\bar{T}$ is not dense.

- (ii) Suppose T is a quotient map. Let $Z = \text{FBfA}[E]/\ker \bar{T}$ and let $Q: \text{FBfA}[E] \rightarrow Z$ be the quotient map. Recall that Z is a Banach f -algebra and that Q is a lattice-algebra homomorphism. By standard factorization arguments there exists an injective and contractive operator $S: Z \rightarrow \text{FBfA}[F]$ such that $\bar{T} = SQ$. Let us show that S is also a lattice-algebra homomorphism. Fix $z_1, z_2 \in Z$ and let $x_1, x_2 \in \text{FBfA}[E]$ be such that $Qx_i = z_i$ for $i = 1, 2$. Since both $\bar{T}$ and Q are lattice-algebra homomorphisms:

$$S|z_1| = S|Qx_1| = SQ|x_1| = \bar{T}|x_1| = |\bar{T}x_1| = |Sz_1|,$$

and also

$$\begin{aligned} S(z_1 z_2) &= S(Qx_1 Qx_2) = SQ(x_1 x_2) \\ &= \bar{T}(x_1 x_2) = \bar{T}(x_1) \bar{T}(x_2) = Sz_1 Sz_2. \end{aligned}$$

Since $\ker T \subseteq \ker Q\eta_E$, $Q\eta_E$ is contractive and T is a quotient map, there exists a contractive operator $R: F \rightarrow Z$ such that $Q\eta_E = RT$. Let $\hat{R}: \text{FBfA}[F] \rightarrow Z$ be its canonical extension. Fix $y \in F$ and let $x \in E$ be such that $Tx = y$. Then

$$S\hat{R}\eta_F(y) = SRy = SRTx = SQ\eta_E x = \bar{T}\eta_E x = \eta_F y.$$

Since $S\hat{R}$ is a lattice-algebra homomorphism, it must be the identity. This shows that S is surjective, and therefore an isometry with inverse $\hat{R}$. In particular, $\bar{T} = SQ$ is a quotient map.

- (iii) Suppose $\bar{T}$ is bijective. It is obvious that T is injective and, by (i), T also has dense range. Let $y \in F$ and let (x_n) be a sequence in E such that $Tx_n \rightarrow y$. Then $\bar{T}(\eta_E(x_n)) \rightarrow \eta_F(y)$ in $\text{FBfA}[F]$. Since $\bar{T}$ is surjective, there exists $f \in \text{FBfA}[E]$ for which $\bar{T}f = \eta_F(y)$. From applying $\bar{T}^{-1}$ to $\bar{T}f = \lim \bar{T}(\eta_E(x_n))$ it follows that $\eta_E(x_n) \rightarrow f$ in $\text{FBfA}[E]$. Hence $(\eta_E(x_n))$ is a Cauchy sequence and, since η_E is an isometric embedding, so is (x_n) . If x denotes the limit of (x_n) in E , then $Tx = y$.
- (iv) Suppose $T^*(B_{F^*})$ is not dense in B_{E^*} . Let $U \subseteq B_{E^*}$ be a weak* open set disjoint from $T^*(B_{F^*})$. By Lemma 8.88, there exists a non-zero $f \in \text{FNfA}[E]$ such that $\text{supp } \iota_E(f) \subseteq U$. Using Lemma 8.92 one gets that, for every $y^* \in B_{F^*}$, $\hat{\iota}_F(\bar{T}f)(y^*) = \hat{\iota}_E(f)(T^*y^*) = 0$. Hence $\bar{T}f = 0$ and $\bar{T}$ is not injective.

Conversely, suppose $T^*(B_{F^*})$ is dense in B_{E^*} . Let $f \in \text{FBfA}[E]$ be such that $\bar{T}f = 0$. Using again the formula from Lemma 8.92,

$$0 = \hat{\iota}_F(\bar{T}f)(y^*) = \hat{\iota}_E(f)(T^*y^*)$$

for every $y^* \in B_{F^*}$. Thus $\hat{\iota}_E(f)$ vanishes on $T^*(B_{F^*})$ and, since it is continuous, the density assumption implies that it vanishes on B_{E^*} . Hence $f = 0$. $\square$

Example 8.93 shows that the converse implication in (iii) above is not true. However, two properties remain to be clarified, namely:

Question 8.95. (i) If $\bar{T}$ is a quotient map, is T also a quotient map?

(ii) Does $\bar{T}$ surjective imply that T is surjective? What about the converse?

Isometries and isometric embeddings are treated in the next section.

8.5.4 Isometries and isometric embeddings

It is direct from the universal property that two isometric Banach spaces have isometric free Banach f -algebras. Do isometric free Banach f -algebras correspond to isometric Banach spaces? The goal of this section is to show that they do when the Banach spaces have smooth dual. The proof follows the same steps as Theorem 1.40, but one has to proceed with more care: every isometry between the free Banach f -algebras induces a w^* -continuous function between the unit balls of the duals that need not be positively homogeneous. Therefore this function cannot be extended to the whole dual, and the results in [IT20] cannot be applied directly. Instead, a refinement of the result in that paper is needed.

Recall that a Banach space X is said to be *smooth* if for every $x \in X$ there exists a unique $f_x \in S_{X^*}$ such that $f_x(x) = \|x\|$. This functional f_x is called the *support functional* at x and is given by

$$f_x(y) = \lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}.$$

The quantity $[x, y] = \|y\|f_y(x)$ defines a semi-inner product with $\|x\| = [x, x]^{1/2}$. We say that x is *orthogonal* to y in the sense of Birkhoff–James, and denote it by $x \perp_{\text{BJ}} y$, if $[y, x] = 0$. Note that this orthogonality is right additive. Moreover, it coincides with the original Birkhoff–James definition; that is,

$$x \perp_{\text{BJ}} y \quad \text{if and only if} \quad \|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 8.96. *Let X and Y be smooth normed spaces and let $f: B_X \rightarrow B_Y$ be a surjective mapping satisfying*

$$|[f(x), f(y)]| = |[x, y]| \quad \text{for all } x, y \in B_X.$$

Then there exist a linear surjective isometry $U: X \rightarrow Y$ and a map $\sigma: B_X \rightarrow \{-1, 1\}$ such that $f = \sigma U|_{B_X}$.

Proof. The first step is to show that, for every $x \in B_X$ and $\lambda \in \mathbb{R}$ such that $\lambda x \in B_X$, there exists a scalar $\gamma = \gamma(\lambda, x)$ with $|\gamma| = |\lambda|$ such that $f(\lambda x) = \gamma f(x)$. This γ is defined by the identity

$$\min_{\xi \in \mathbb{R}} \|f(\lambda x) - \xi f(x)\| = \|f(\lambda x) - \gamma f(x)\|.$$

Note in particular that $\|f(\lambda x) - \gamma f(x)\| \leq \|f(\lambda x)\| \leq 1$. By definition

$$\|f(\lambda x) - \gamma f(x) + \mu f(x)\| \geq \|f(\lambda x) - \gamma f(x)\|$$

holds for every $\mu \in \mathbb{R}$, so by Birkhoff–James orthogonality, $f(\lambda x) - \gamma f(x) \perp_{\text{BJ}} f(x)$. Since f is surjective and $f(\lambda x) - \gamma f(x) \in B_Y$, there exists $z \in B_X$ such that $f(z) = f(\lambda x) - \gamma f(x)$. From $f(z) \perp_{\text{BJ}} f(x)$ we get $z \perp_{\text{BJ}} x$ and then $z \perp_{\text{BJ}} \lambda x$ and $f(z) \perp_{\text{BJ}} f(\lambda x)$. Using that the semi-inner product is right additive we conclude that

$$f(z) \perp_{\text{BJ}} (f(\lambda x) - \gamma f(x)).$$

But the right hand side is precisely $f(z)$. Thus $f(z) = 0$ and $f(\lambda x) = \gamma f(x)$. Moreover,

$$|\lambda| \|x\| = \|\lambda x\| = \|f(\lambda x)\| = \|\gamma f(x)\| = |\gamma| \|f(x)\| = |\gamma| \|x\|$$

implies $|\lambda| = |\gamma|$.

Next, let $x, y \in B_X$ be linearly independent and such that $x + y \in B_X$. We want to show that $f(x + y) = \alpha f(x) + \beta f(y)$, where $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ and $|\alpha| = 1 = |\beta|$. As before, define $\alpha, \beta \in \mathbb{R}$ by the relation

$$\min_{\xi, \eta \in \mathbb{R}} \|f(x + y) - \xi f(x) - \eta f(y)\| = \|f(x + y) - \alpha f(x) - \beta f(y)\|$$

and note that $f(x + y) - \alpha f(x) - \beta f(y) \in B_Y$. Then

$$f(x + y) - \alpha f(x) - \beta f(y) \perp_{\text{BJ}} f(x) \quad \text{and} \quad f(x + y) - \alpha f(x) - \beta f(y) \perp_{\text{BJ}} f(y).$$

Take $z \in B_X$ such that $f(z) = f(x + y) - \alpha f(x) - \beta f(y)$. Then $f(z) \perp_{\text{BJ}} f(x)$ implies $z \perp_{\text{BJ}} x$, $f(z) \perp_{\text{BJ}} f(y)$ implies $z \perp_{\text{BJ}} y$ and the right additivity of orthogonality implies $z \perp_{\text{BJ}} x + y$ and $f(z) \perp_{\text{BJ}} f(x + y)$. Hence $f(z) \perp_{\text{BJ}} f(z)$ and $f(z) = 0$. Let us show that $|\alpha| = 1$. Let $\lambda_0 \in \mathbb{R}$ be defined by

$$\|x + \lambda_0 y\| = \min_{\lambda \in \mathbb{R}} \|x + \lambda y\|.$$

Once again, $x + \lambda_0 y \in B_X$. Then $x + \lambda_0 y \perp_{\text{BJ}} y$ and $x + \lambda_0 y \not\perp_{\text{BJ}} x$, for otherwise $x + \lambda_0 y = 0$, contradicting the fact that x and y are linearly independent. Denote $w = x + \lambda_0 y$. Since $w \perp_{\text{BJ}} y$ we also have $f(w) \perp_{\text{BJ}} f(y)$. Then $[f(x + y), f(w)] = \alpha[f(x), f(w)]$ and

$$|\alpha| |[x, w]| = |\alpha| |[f(x), f(w)]| = |[f(x + y), f(w)]| = |[x + y, w]| = |[x, w]|.$$

Since $[x, w] \neq 0$, it follows $|\alpha| = 1$. Similarly, $|\beta| = 1$.

Let us consider the projective spaces $\mathbb{P}X$ and $\mathbb{P}Y$ obtained as respective quotients of X and Y by identifying proportional vectors. Note that the map

$$\begin{aligned} \tilde{f}: \mathbb{P}X &\longrightarrow \mathbb{P}Y \\ \langle x \rangle &\longmapsto \langle f(x/\|x\|) \rangle \end{aligned}$$

is well-defined and surjective. We want to check that it satisfies the assumptions of the Fundamental Theorem of Projective Geometry (see [IT20, Theorem 2.1]) whenever $\dim X \geq 3$.

First we need to check its image is not contained in a projective line. Let $x \in S_X$ be a unit vector. Since $\dim X \geq 3$ we can choose unit vectors $y \in \ker f_x$ and $z \in \ker f_x \cap \ker f_y$. From $x \perp_{\text{BJ}} y$, $x \perp_{\text{BJ}} z$ and $y \perp_{\text{BJ}} z$ it follows that x, y, z are linearly independent. Then $f(x), f(y), f(z)$ are unit vectors such that $f(x) \perp_{\text{BJ}} f(y)$, $f(x) \perp_{\text{BJ}} f(z)$ and $f(y) \perp_{\text{BJ}} f(z)$. As before we conclude that they are linearly independent. So the image of f is not contained in a two-dimensional subspace, thus the image of $\tilde{f}$ is not contained in a projective line.

Second we need to check that $\tilde{f}$ preserves projective lines. Suppose $c = \lambda a + \mu b$ with $a, b, c \neq 0$. Without loss of generality we may assume $a, b \in S_X$. Set $M = \max\{|\lambda|, |\mu|, \|c\|\}$. Then

$$\begin{aligned} \tilde{f}(\langle c \rangle) &= \langle f(\frac{c}{\|c\|}) \rangle \\ &= \langle f(\frac{M}{\|c\|} \frac{c}{M}) \rangle \\ &= \langle f(\frac{\lambda}{M} a + \frac{\mu}{M} b) \rangle \end{aligned}$$

where in the third equality we have used that both c/M and $c/\|c\|$ belong to B_X . After the previous computation, using the properties of f together with

the fact that $a, b, \lambda/Ma, \mu/Mb \in B_X$, it follows that $\tilde{f}(\langle c \rangle)$ is contained in the projective line that passes through $\langle a \rangle$ and $\langle b \rangle$.

By the Fundamental Theorem of Projective Geometry there exists a bijective linear map $A: X \rightarrow Y$ such that

$$\tilde{f}(\langle x \rangle) = \langle Ax \rangle \quad \text{for all } x \in X.$$

This means in particular that for every $x \in S_X$ there exists a nonzero $\lambda(x) \in \mathbb{R}$ such that $f(x) = \lambda(x)Ax$. More generally, for nonzero $x \in B_X$, since $f(x/\|x\|) = \gamma f(x)$, one also has $f(x) = \lambda(x)Ax$ for some $\lambda(x) \neq 0$. Note that, if $x \in B_X$ and $\mu \in \mathbb{R}$ are nonzero, and $\mu x \in B_X$, then

$$\gamma \lambda(x)Ax = \gamma f(x) = f(\mu x) = \lambda(\mu x)\mu Ax$$

and therefore

$$|\lambda(\mu x)| = |\gamma/\mu| |\lambda(x)| = |\lambda(x)|.$$

Let $x, y \in B_X$ be linearly independent and such that $x + y \in B_X$. Then

$$\lambda(x + y)A(x + y) = f(x + y) = \alpha f(x) + \beta f(y) = \alpha \lambda(x)Ax + \beta \lambda(y)Ay$$

and since Ax and Ay are linearly independent, it follows that

$$|\lambda(x)| = |\alpha \lambda(x)| = |\lambda(x + y)| = |\beta \lambda(y)| = |\lambda(y)|.$$

In conclusion, $|\lambda(x)|$ is constantly $|\lambda|$. By letting $\sigma(x) = \lambda(x)/|\lambda|$ and $U = |\lambda|A$ one gets the desired result for $\dim X \geq 3$.

For $\dim X = 2$, let $x_0, y_0 \in X$ be such that $x_0 \perp_{BJ} y_0$ and $\|x_0\| = \|y_0\| = 1/2$. For every μ with $\|x_0 + \mu y_0\| \leq 1$ (in particular, for every $\mu \in [-1, 1]$), there exist $\omega_1, \omega_2 \in \mathbb{R}$ such that

$$f(x_0 + \mu y_0) = \omega_1 f(x_0) + \omega_2 f(y_0)$$

with $|\omega_1| = 1$ and $|\omega_2| = |\mu|$.

Define $\omega(\mu) = \omega_1$ and $h(\mu) = \omega_2/\omega_1$. Once we show that $h(\mu) = \mu h(1)$ for all $\mu \in [-1, 1]$, most of the work will be done. For every $\lambda, \mu \in [-1, 1]$ with $\|x_0 + (\lambda + \mu)y_0\| \leq 1$ we have

$$f(x_0 + (\lambda + \mu)y_0) = \omega(\lambda + \mu)(f(x_0) + h(\lambda + \mu)f(y_0))$$

but also

$$\begin{aligned} f(x_0 + (\lambda + \mu)y_0) &= f((x_0 + \lambda y_0) + \mu y_0) \\ &= \omega'_4 f(x_0 + \lambda y_0) + \omega_3 f(y_0) \\ &= \omega_4 f(x_0) + \omega_4 h(\lambda) f(y_0) + \omega_3 f(y_0) \end{aligned}$$

for certain real numbers $\omega_3, \omega'_4, \omega_4$ with $|\omega_3| = |\mu|$, $|\omega'_4| = 1$ and $|\omega_4| = 1$. Since $f(x_0)$ and $f(y_0)$ are linearly independent, $\omega_4 = \omega(\lambda + \mu)$ and

$$\omega(\lambda + \mu)h(\lambda) + \omega_3 = \omega(\lambda + \mu)h(\lambda + \mu)$$

which implies

$$h(\lambda + \mu) = h(\lambda) + \frac{\omega_3}{\omega}. \quad (8.6)$$

Taking absolute values, we have

$$|\lambda + \mu| = \left| h(\lambda) + \frac{\omega_3}{\omega} \right|.$$

Since $|\omega_3/\omega| = |\mu|$, if $\mu \neq 0$, then we can divide by $|\omega_3/\omega|$ to get

$$\left| \frac{\lambda}{\mu} + 1 \right| = \left| h(\lambda) \frac{\omega}{\omega_3} + 1 \right|.$$

Using that $|h(\lambda)| = |\lambda|$ it easily follows

$$\frac{\lambda}{\mu} = h(\lambda) \frac{\omega}{\omega_3}. \quad (8.7)$$

Hence, if also $\lambda \neq 0$, then we can put together (8.6) and (8.7) to get

$$h(\lambda + \mu) = h(\lambda) \left(1 + \frac{\mu}{\lambda} \right).$$

Fix $0 \leq \eta < 1$. In this case, $\|x_0 + \eta y_0\| \leq 1$. We may set $\lambda = 1$ and $\mu = \eta - 1$ in previous equation to get $h(\eta) = \eta h(1)$. Similarly, $\lambda = -1$ and $\mu = 1 - \eta$ yield $h(-\eta) = \eta h(-1)$. Finally, setting $\lambda = 1/2$ and $\mu = -1$ yields $h(-1/2) = h(1/2)$, which in turn implies $h(-1) = h(1)$. Hence $h(\mu) = \mu h(1)$ holds for all $\mu \in [-1, 1]$; that is,

$$f(x_0 + \mu y_0) = \omega(\mu)(f(x_0) + \mu h(1)f(y_0)),$$

where $|\omega(\mu)| = 1 = |h(1)|$.

By the same argument, for every $\lambda \in [-1, 1]$ there exist signs $\omega'(\lambda)$ and $g(1)$ such that

$$f(\lambda x_0 + y_0) = \omega'(\lambda)(\lambda g(1)f(x_0) + f(y_0)).$$

Since

$$\omega'(1)(g(1)f(x_0) + f(y_0)) = f(x_0 + y_0) = \omega(1)(f(x_0) + h(1)f(y_0))$$

we conclude that $h(1)g(1) = 1$, that is, $h(1) = g(1)$.

Define a linear map $U: X \rightarrow Y$ by $Ux_0 = f(x_0)$ and $Uy_0 = h(1)f(y_0)$. We are going to show that f coincides with $U|_{B_X}$ up to a sign. Every element of the unit ball can be written uniquely as $\lambda x_0 + \mu y_0$ for certain $\lambda, \mu \in \mathbb{R}$. Suppose first that $|\lambda| \geq |\mu|$. If $\lambda = 0$, then $f(0) = 0 = U(0)$, so assume also $\lambda \neq 0$. Then $x_0 + \mu/\lambda y_0 \in B_X$, and therefore there exists a $\gamma \in \mathbb{R}$ with $|\gamma| = |\lambda|$ such that

$$\begin{aligned} f(\lambda x_0 + \mu y_0) &= \gamma f(x_0 + \mu/\lambda y_0) \\ &= \gamma \omega(\mu/\lambda)(f(x_0) + \mu/\lambda h(1)f(y_0)) \\ &= \gamma \omega U(x_0 + \mu/\lambda y_0) \\ &= \frac{\gamma}{\lambda} \omega U(\lambda x_0 + \mu y_0). \end{aligned}$$

Hence f coincides with U at $\lambda x_0 + \mu y_0$ up to a sign $\omega\gamma/\lambda$. Similarly, if $|\mu| > |\lambda|$, then there exists a $\delta \in \mathbb{R}$ with $|\delta| = |\mu|$ such that

$$\begin{aligned} f(\lambda x_0 + \mu y_0) &= \delta f(\lambda/\mu x_0 + y_0) \\ &= \delta \omega'(\lambda/\mu)(\lambda/\mu g(1)f(x_0) + f(y_0)) \\ &= \delta \frac{\omega'}{\mu} g(1)(\lambda f(x_0) + \mu h(1)f(y_0)) \\ &= \delta \frac{\omega'}{\mu} g(1)U(\lambda x_0 + \mu y_0), \end{aligned}$$

and again f coincides with U at $\lambda x_0 + \mu y_0$ up to a sign $\delta\omega'/\mu g(1)$.

It only remains to check that U preserves the norm. An element of X can be written uniquely as $\lambda x_0 + \mu y_0$. Suppose first that $|\lambda| > |\mu|$, so as to have $\|x_0 + \mu/\lambda y_0\| \leq 1$. In this case,

$$\begin{aligned} \|\lambda x_0 + \mu y_0\| &= |\lambda| \|x_0 + \mu/\lambda y_0\| \\ &= |\lambda| \|f(x_0 + \mu/\lambda y_0)\| \\ &= |\lambda| \|U(x_0 + \mu/\lambda y_0)\| \\ &= \|U(\lambda x_0 + \mu y_0)\|. \end{aligned}$$

The case $|\mu| \geq |\lambda|$ is completely analogous. This completes the proof when $\dim X = 2$.

When $\dim X = 1$, fix a unit vector $x_0 \in X$, and define a linear map $U: X \rightarrow Y$ by $U(\lambda x_0) = \lambda f(x_0)$. This map clearly preserves the norm. For every $\lambda \in [-1, 1]$, $\lambda \neq 0$, there exists $\gamma \in \mathbb{R}$ with $|\gamma| = |\lambda|$ such that

$$f(\lambda x_0) = \gamma f(x_0) = \gamma/\lambda U(\lambda x_0).$$

Hence, f coincides with U at λx_0 up to a sign γ/λ . □

Remark 8.97. The previous theorem is still true if $f: \delta B_X \rightarrow \delta B_Y$, for some $\delta > 0$, and $f(\delta B_X) = \delta B_Y$. Indeed, let X_δ (resp. Y_δ) be the space X (resp. Y) with the norm scaled by $1/\delta$. These are still smooth normed spaces satisfying $B_{X_\delta} = \delta B_X$ and $B_{Y_\delta} = \delta B_Y$. Moreover, one can easily check that the semi-inner products are related by $[x, y]_{X_\delta} = [x, y]_X/\delta^2$, and the same is true for Y and Y_δ . Hence $f: B_{X_\delta} \rightarrow B_{Y_\delta}$ is a surjective mapping that preserves the semi-inner product. By the previous theorem, there exist a linear surjective isometry $U: X_\delta \rightarrow Y_\delta$ and a map $\sigma: B_{X_\delta} \rightarrow \{-1, 1\}$ such that $f = \sigma U|_{B_{X_\delta}}$. It is then clear that $U: X \rightarrow Y$ is still a linear surjective isometry, and that $f = \sigma U|_{\delta B_X}$.

With this we can now state and prove our main theorem in this section. The proof follows closely that of Theorem 1.40.

Theorem 8.98. *Let E and F be Banach spaces with smooth dual. An operator $T: \text{FBfA}[E] \rightarrow \text{FBfA}[F]$ is a surjective lattice-algebra isometry if and only if $T = \bar{V}$, for some surjective isometry $V: E \rightarrow F$. In particular, E and F are isometric if and only if $\text{FBfA}[E]$ and $\text{FBfA}[F]$ are lattice-algebra isometric.*

Proof. First we are going to construct a map $\phi_T: B_{F^*} \rightarrow B_{E^*}$. Let $y^* \in B_{F^*}$. Extend $y^*: F \rightarrow \mathbb{R}$ to a unique contractive lattice-algebra homomorphism $\widehat{y^*}: \text{FBfA}[F] \rightarrow \mathbb{R}$. Define $\phi_T(y^*) = T^*(\widehat{y^*}) \circ \eta_E$ (that is, $\widehat{\phi_T(y^*)} = T^*(\widehat{y^*})$).

We will use the lattice-algebraic homomorphisms $\hat{\iota}_E: \text{FBfA}[E] \rightarrow C(B_{E^*})$ and $\hat{\iota}_F: \text{FBfA}[F] \rightarrow C(B_{F^*})$ to express ϕ_T in another way. Note that, if $g \in \text{FBfA}[F]$, then $\widehat{y^*}(g) = \hat{\iota}_F(g)(y^*)$. Indeed, one can easily check that $g \mapsto \hat{\iota}_F(g)(y^*)$ is a contractive lattice-algebra homomorphism extending y^* . If $f \in \text{FBfA}[E]$ and $y^* \in B_{F^*}$, then

$$(\hat{\iota}_E(f) \circ \phi_T)(y^*) = \hat{\iota}_E(f)(\phi_T(y^*)) = \widehat{\phi_T(y^*)}(f) = T^*(\widehat{y^*})(f) = \hat{\iota}_F(Tf)(y^*).$$

Thus $\hat{\iota}_F(Tf) = \hat{\iota}_E(f) \circ \phi_T$. Plugging in $f = \eta_E(x)$ yields $\phi_T(y^*)(x) = \hat{\iota}_F(T\eta_E(x))(y^*)$ for all $y^* \in B_{F^*}$. In particular, if (y_α^*) is a net that weak* converges to y^* in B_{F^*} , then

$$\phi_T(y_\alpha^*)(x) = \hat{\iota}_F(T\eta_E(x))(y_\alpha^*) \rightarrow \hat{\iota}_F(T\eta_E(x))(y^*) = \phi_T(y^*)(x).$$

It follows that ϕ_T is weak* to weak* continuous. It is also direct to check that ϕ_T is invertible with $\phi_T^{-1} = \phi_{T^{-1}}$.

Suppose that B_{E^*} and B_{F^*} are equipped with the corresponding semi-inner products. We are going to show that ϕ_T preserves the semi-inner product when restricted to $\frac{1}{2}B_{F^*}$. According to [OTT24, Lemma 10.19], showing that ϕ_T preserves the semi-inner product is equivalent to showing that

$$\max_{\pm} \|\phi_T(x^*) \pm \phi_T(y^*)\| = \max_{\pm} \|x^* \pm y^*\|$$

holds for every $x^*, y^* \in \frac{1}{2}B_{F^*}$. Consider the operator

$$\begin{aligned} S: \quad F &\longrightarrow \ell_1^2 \\ y &\longmapsto (x^*(y), y^*(y)) \end{aligned} ,$$

where ℓ_1^2 is equipped with the pointwise product. It is direct to check that $\|S\| = \max_{\pm} \|x^* \pm y^*\| \leq 1$. One can also check that its extension $\hat{S}: \text{FBfA}[F] \rightarrow \ell_1^2$ is $\hat{S}(f) = (\hat{\iota}_F(f)(x^*), \hat{\iota}_F(f)(y^*))$. Then $\hat{S}T: \text{FBfA}[E] \rightarrow \ell_1^2$ is a lattice-algebra homomorphism with $\|\hat{S}T\| = \|\hat{S}\|$. Moreover, the universal property of the free Banach f -algebra implies $\|\hat{S}T\eta_E\| = \|\hat{S}T\|$. Unfolding the definitions:

$$\hat{S}T(\eta_E(x)) = (\hat{\iota}_F((T\eta_E)(x))(x^*), \hat{\iota}_F((T\eta_E)(x))(y^*)) = (\phi_T(x^*)(x), \phi_T(y^*)(x)).$$

By direct computation, $\|\hat{S}T\eta_E\| = \max_{\pm} \|\phi_T(x^*) \pm \phi_T(y^*)\|$, and this coincides with $\|S\| = \max_{\pm} \|x^* \pm y^*\|$.

For the same reasons, $\phi_{T^{-1}}$ preserves the semi-inner product when restricted to $\frac{1}{2}B_{E^*}$. This implies that $\phi_T|_{\frac{1}{2}B_{F^*}}: \frac{1}{2}B_{F^*} \rightarrow \frac{1}{2}B_{E^*}$ is surjective. Hence we can apply Theorem 8.96 and the remark after it to this map: there exists a linear surjective isometry $U: F^* \rightarrow E^*$ and a map $\sigma: \frac{1}{2}B_{F^*} \rightarrow \{-1, 1\}$ such that $\phi_T = \sigma U|_{\frac{1}{2}B_{F^*}}$. We claim that σ is continuous on $\frac{1}{2}B_{F^*} \setminus \{0\}$ for the norm topology. Suppose it was not. Then there would exist a convergent sequence

$y_k^* \rightarrow y^*$ in $\frac{1}{2}B_{F^*} \setminus \{0\}$ such that $\sigma(y_k^*)$ does not converge to $\sigma(y^*)$. To fix ideas, suppose $\sigma(y^*) = -1$ (the other case is identical). Then there exists a subsequence of y_k^* , which we denote the same way, for which $\sigma(y_k^*) = 1$. Note that $Uy_k^* \rightarrow Uy^*$. At the same time, since norm convergence implies weak* convergence, $Uy_k^* = \phi_T(y_k^*) \rightarrow \phi_T(y^*) = -Uy^*$. Hence $Uy^* = 0$, a contradiction with the fact that $y^* \neq 0$.

When $\dim F > 1$, $\frac{1}{2}B_{F^*} \setminus \{0\}$ is connected, and therefore σ , being continuous, must be constant. Absorbing σ in U , we have $\phi_T = U|_{\frac{1}{2}B_{F^*}}$, and therefore that U is weak* continuous on bounded sets. This implies that U is an adjoint operator; its pre-adjoint V is an isometry such that $T = \bar{V}$ (the details are the same as in the end of the proof of [OTTT24, Theorem 10.18]). When $\dim F = 1$, but $\dim E > 1$, we can use the same argument with $\phi_{T^{-1}}$.

It only remains to consider the case $\dim E = \dim F = 1$. Recall from Theorem 8.56 that, in this case, $\text{FBfA}[E]$ is isomorphic to $C([0, 1] \times \{-1, 1\})$ with the supremum norm, pointwise lattice structure, and product

$$(f \star g)(r, u) = rf(r, u)g(r, u)$$

for every $f, g \in C([0, 1] \times \{-1, 1\})$ and $(r, u) \in [0, 1] \times \{-1, 1\}$. The canonical map $\eta: \mathbb{R} \rightarrow C([0, 1] \times \{-1, 1\})$ is determined by $\eta(1)(r, u) = u$. Suppose

$$T: C([0, 1] \times \{-1, 1\}) \rightarrow C([0, 1] \times \{-1, 1\})$$

is a surjective lattice-algebra isometry. Being a lattice isometry, it must be of the form $Tf = f \circ \phi$, where $\phi: [0, 1] \times \{-1, 1\} \rightarrow [0, 1] \times \{-1, 1\}$ is a homeomorphism ([AA02a, Theorem 4.25]). Put $\phi(r, u) = (\phi_1(r, u), \phi_2(r, u))$. Since T is also an algebra homomorphism,

$$T(fg)(r, u) = \phi_1(r, u)f(\phi(r, u))g(\phi(r, u))$$

must be equal to

$$(TfTg)(r, u) = rf(\phi(r, u))g(\phi(r, u))$$

for every $f, g \in C([0, 1] \times \{-1, 1\})$. Thus $\phi_1(r, u) = r$. Being ϕ_2 continuous, it must be constant on connected components. That is, there exists $\sigma \in \{-1, 1\}$ such that $\phi_2(r, u) = \sigma u$. The isometry

$$\begin{aligned} V: \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sigma x \end{aligned}$$

is such that

$$T(\eta_x)(r, u) = \eta_x(\phi(r, u)) = x\phi_2(r, u) = x\sigma u = \eta_{Vx}(r, u).$$

Hence $T = \bar{V}$, as wanted. $\square$

What can be said about isometric embeddings? In the case of free Banach lattices, the problem of whether an embedding between two Banach spaces becomes an embedding between the free Banach lattices is reduced to a certain

Banach space question involving extensions of operators to L_1 (see [OTTT24, Theorem 3.7]). In one direction, something similar can be said about free Banach f -algebras. But now, instead of considering operators with range in L_1 , we have to consider operators with range in arbitrary Banach f -algebras.

More precisely, suppose that $i: E \rightarrow F$ is an isometric embedding between Banach spaces, and that for every contractive operator $T_E: E \rightarrow A_E$, where A_E is a Banach f -algebra, there exists a contractive operator $T_F: F \rightarrow A_F$, where A_F is a Banach f -algebra in which A_E embeds isometrically through a lattice-algebra homomorphism $j: A_E \rightarrow A_F$, making the diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & F \\ T_E \downarrow & & \downarrow T_F \\ A_E & \xrightarrow{j} & A_F \end{array} \quad (8.8)$$

commute. Then the map

$$\bar{i}: \text{FBfA}[E] \rightarrow \text{FBfA}[F]$$

is an isometric embedding. Indeed, fix $f \in \text{FBfA}[E]$ and $\varepsilon > 0$, and let $T_E: E \rightarrow A_E$ be a contractive operator, where A_E is a Banach f -algebra, satisfying $\|\hat{T}_E f\| > \|f\| - \varepsilon$. Let $T_F: F \rightarrow A_F$ and $j: A_E \rightarrow A_F$ be as in (8.8). In the diagram

$$\begin{array}{ccc} \text{FBfA}[E] & \xrightarrow{\bar{i}} & \text{FBfA}[F] \\ \hat{T}_E \left(\begin{array}{ccc} \uparrow \eta_E & & \uparrow \eta_F \\ E & \xrightarrow{i} & F \\ \downarrow T_E & & \downarrow T_F \end{array} \right) \hat{T}_F & & \\ A_E & \xrightarrow{j} & A_F \end{array}$$

we have that

$$\hat{T}_F \bar{i} \eta_E = \hat{T}_F \eta_F i = T_F i = j T_E = j \hat{T}_E \eta_E.$$

By the universal property of $\text{FBfA}[E]$, it follows that $\hat{T}_F \bar{i} = j \hat{T}_E$. Hence

$$\|\bar{i} f\| \geq \|\hat{T}_F \bar{i} f\| = \|j \hat{T}_E f\| = \|\hat{T}_E f\| > \|f\| - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\|\bar{i} f\| \geq \|f\|$, and therefore $\bar{i}$ is an isometric embedding.

Question 8.99. Suppose $\bar{i}: \text{FBfA}[E] \rightarrow \text{FBfA}[F]$ is an isometric embedding. Is it true that for every contractive map $T_E: E \rightarrow A_E$, where A_E is a Banach f -algebra, there exists a contractive operator $T_F: F \rightarrow A_F$, where A_F is a Banach f -algebra in which A_E embeds isometrically through j , making the diagram (8.8) commute?

8.5.5 Free Banach f -algebras with identity

In this final section we discuss free objects in the category of (Archimedean, normed, Banach) f -algebras with an algebraic identity. We always assume that the homomorphisms in these categories preserve the identity.

Recall from Section 1.4 that the identity of an Archimedean f -algebra is positive and a weak order unit, and that every Banach f -algebra with identity can be renormed to have an identity of norm one. From now on, we shall assume that Banach f -algebras with identity have identities of norm one. In this case, every such Banach f -algebra is lattice-algebra isometric to $C(K)$, for some compact Hausdorff space K (this is an immediate consequence of Theorem 3.1). Hence, the category of Banach f -algebras with identity is precisely that of $C(K)$ -spaces, where the morphisms are lattice-algebra homomorphisms preserving the algebraic identity. This, in turn, is the category of AM-spaces with unit whose morphisms are the lattice homomorphisms that preserve the norming unit of the AM-space.

Let us now define precisely the object we are trying to construct.

Definition 8.100. Let E be a vector space. The *free Archimedean f -algebra with identity generated by E* is an Archimedean f -algebra $\text{FA}fA^1[E]$ with identity 1 together with a linear map $\delta_E: E \rightarrow \text{FA}fA^1[E]$ such that, for every Archimedean f -algebra A with identity 1_A and every linear map $T: E \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FA}fA^1[E] \rightarrow A$ satisfying $\hat{T}(1) = 1_A$ and $\hat{T}\delta_E = T$.

To construct the free Archimedean f -algebra we used LLA expressions. These were formal expressions obtained from finitely many formal variables using: a 0-ary operation 0, a unary operation λ for every $\lambda \in \mathbb{R}$, and three binary operations $+$, $\vee$ and $\cdot$. If we add an additional 0-ary operation 1, we obtain expressions that can be evaluated at any lattice-ordered algebra with identity. These we shall call *1-LLA expressions*.

The following is a simple consequence of Theorem 1.30. The proof is essentially the same as that of Lemma 8.3.

Lemma 8.101. *Let Φ be a 1-LLA expression. If Φ vanishes on $\mathbb{R}$, then it also vanishes on every Archimedean f -algebra with identity.*

With this result, the construction of $\text{FA}fA^1[E]$ follows the same steps as that of $\text{FA}fA[E]$ presented in Proposition 8.46.

Proposition 8.102. *Let E be a vector space. The free Archimedean f -algebra with identity generated by E is*

$$\text{FA}fA^1[E] = \text{VLA}(\{\delta_x : x \in E\} \cup \{1\}) \subseteq \mathbb{R}^{E^\#},$$

where $1(\omega) = 1$ for all $\omega \in E^\#$, together with the map $\delta_E: E \rightarrow \text{FA}fA^1[E]$, $\delta_E(x) = \delta_x$.

At this point we could proceed as in Section 8.3.1 to obtain an abstract construction of the free Banach f -algebra with identity. But, as noted above, this object is the same as the free AM-space with unit, which was already considered in [JSLT⁺22, Theorem 5.4].

Theorem 8.103 ([JSLT⁺22, Theorem 5.4]). *Let E be a Banach space. Let $\phi_E: E \rightarrow C(B_{E^*})$ be the natural embedding. For every compact Hausdorff space K and every norm one operator $T: E \rightarrow C(K)$ there exists a unique lattice homomorphism $\hat{T}: C(B_{E^*}) \rightarrow C(K)$ satisfying $\hat{T} \circ \phi_E = T$ and $\hat{T}\mathbb{1}_{B_{E^*}} = \mathbb{1}_K$. Moreover, $\|\hat{T}\| = 1$ and $\hat{T}$ is an algebra homomorphism.*

The free Banach f -algebra with identity generated by E is not only contained in $C(B_{E^*})$, but is in fact the whole $C(B_{E^*})$, and the free norm is the supremum norm. Note also that, if we want to preserve the norm, we can only extend norm one operators, since every extension has norm one (because it sends the identity to the identity).

Free objects in categories of f -algebras are not exhausted by the Banach setting. Working instead in the category of uniformly complete Archimedean f -algebras, the free object over a finite set of generators turns out to be the space of continuous polynomial growth functions on $\mathbb{R}^n$, and its universal property is precisely what is usually called polynomial growth continuous function calculus. In Chapter 9 we study this function calculus and characterize the Archimedean lattice-ordered algebras with identity that admit it.

9 POLYNOMIAL GROWTH CONTINUOUS FUNCTION CALCULUS

David Muñoz-Lahoz, *Lattice-ordered algebras admitting a polynomial growth continuous function calculus*, Preprint, arXiv:2604.20294, 2026

In [LT20], N. J. Laustsen and V. G. Troitsky characterized the Archimedean vector lattices that admit a positively homogeneous function calculus. More precisely, they showed that if X is an Archimedean vector lattice, and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ for some $n \in \mathbb{N}$, then the following are equivalent:

- There is a vector lattice homomorphism $\Phi_{\mathbf{x}}: H_n \rightarrow X$ with $\Phi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$, where H_n denotes the vector lattice of positively homogeneous continuous functions defined on $\mathbb{R}^n$, and $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$ is the i -th coordinate projection.
- There exists $f \in X$ such that $f \geq |x_1| \vee \dots \vee |x_n|$ and the norm

$$\|x\|_f = \inf\{\lambda \in [0, \infty) : |x| \leq \lambda f\},$$

defined on the order ideal I_f generated by f , is complete when restricted to the closed sublattice generated by $x_1, \dots, x_n$ in I_f .

The goal of this chapter is to characterize, in a similar vein, the Archimedean lattice-ordered algebras with identity that admit a polynomial growth continuous function calculus. More precisely, if X is a lattice-ordered algebra with identity 1_X , denote by PG_n the lattice-ordered algebra formed by the continuous functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ for which there exist $N \in \mathbb{N}$ and $M \in \mathbb{R}_+$ such that $|f| \leq M(1 + |\pi_1| + \dots + |\pi_n|)^N$ (where $1: \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the function that is identically one). Then the following are equivalent:

- There is a lattice-algebra homomorphism $\Psi_{\mathbf{x}}: PG_n \rightarrow X$ with $\Psi_{\mathbf{x}}(1) = 1_X$ and $\Psi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$.
- There exist $f \geq 1_X \vee |x_1| \vee \dots \vee |x_n|$ and an f -subalgebra Y of X such that $1_X, x_1, \dots, x_n \in Y$ and, for every $m \in \mathbb{N}$, the norm $\|\cdot\|_{f^m}$ is complete on $Y \cap I_{f^m}$.

Compared to the Archimedean vector lattice setting, the condition at hand is notably more intricate. The extra algebraic layer (and the passage to algebra homomorphisms) is not really what drives this added complexity; the real source

is that PG_n has a richer internal structure than H_n . Recall that H_n is an AM-space with unit, a feature that keeps the corresponding conditions tidy. While PG_n itself fails to be an AM-space with unit, it does decompose as a countable union of such spaces. Concretely, setting $d = \mathbb{1} \vee |\pi_1| \vee \cdots \vee |\pi_n|$, one has

$$PG_n = \bigcup_{m \in \mathbb{N}} I_{d^m}, \quad \text{with} \quad I_{d^m} \subseteq I_{d^{m+1}}.$$

This particular structure will be central in proving that the two conditions above are equivalent. The appearance of an f -subalgebra in the condition is also expected, since f -algebras are the only lattice-ordered algebras admitting a polynomial growth continuous function calculus (Proposition 9.4).

As noted in [LT20], positively homogeneous continuous function calculus plays a fundamental role in the theory of Banach lattices. This form of function calculus was first introduced by Yudin [Yud39] and Krivine [Kri74] for Banach lattices, and was subsequently generalized to uniformly complete Archimedean vector lattices in a seminal paper by Buskes, de Pagter and van Rooij [BdPvR91]. In that work, the authors deal not only with uniformly complete Archimedean vector lattices, but also with uniformly complete Archimedean f -algebras. It is in the latter context that the existence of a polynomial growth continuous function calculus is established. Hence, just as Laustsen and Troitsky did for Archimedean vector lattices, it seems relevant to provide a useful characterization of the Archimedean lattice-ordered algebras that admit such a function calculus.

Section 9.1 provides two further arguments to justify why polynomial growth continuous function calculus is the natural function calculus to consider in lattice-ordered algebras. The first shows that this function calculus is the natural extension of positively homogeneous continuous function calculus on vector lattices. The second relies on viewing function calculus through the lens of free objects. Although the latter is an approach certainly known and used by experts, we have not found it in the literature. As part of this approach, in Theorem 9.8 we characterize the free object generated by finitely many elements in the category of uniformly complete Archimedean f -algebras.

Section 9.2 contains the main result of this chapter, Theorem 9.10, which provides the equivalent conditions described at the beginning of this introduction. As a consequence, we obtain in Corollary 9.17 a characterization of the Archimedean lattice-ordered algebras with identity admitting a polynomial growth continuous function calculus. We also use examples to illustrate the different ways in which a lattice-ordered algebra may fail to admit such a function calculus.

Finally, in Section 9.3, we prove that if a vector space admits a sufficiently strong polynomial growth continuous function calculus (without constants), then it is automatically a commutative f -algebra. This result is an analogue of [LT20, Theorem 1.4].

9.1 WHY POLYNOMIAL GROWTH CONTINUOUS FUNCTION CALCULUS?

The goal of this section is to introduce polynomial growth continuous function calculus in lattice-ordered algebras with identity and to justify, in two ways, why it is the natural function calculus to consider in this category. We also show that every lattice-ordered algebra having such a function calculus must be an f -algebra.

9.1.1 Polynomial growth continuous function calculus as a natural extension of positively homogeneous continuous function calculus

A real-valued function f defined on a vector space X is said to be *positively homogeneous* if, for every $x \in X$ and every $\lambda \in \mathbb{R}_+$, the identity $f(\lambda x) = \lambda f(x)$ holds. For $n \in \mathbb{N}$, denote by H_n the vector lattice of positively homogeneous continuous functions defined on $\mathbb{R}^n$. It is standard that H_n is lattice isomorphic to $C([-1, 1]^n)$, the isomorphism being the map that sends a function to its restriction to $[-1, 1]^n$. Pulling up the norm from $C([-1, 1]^n)$, H_n becomes a Banach lattice. In particular, H_n is uniformly complete.

For $i \in \{1, \dots, n\}$, the i -th coordinate projection

$$\begin{aligned} \pi_i: \quad \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (t_1, \dots, t_n) &\longmapsto t_i \end{aligned}$$

belongs to H_n . We shall denote the i -th coordinate projection by π_i regardless of which $\mathbb{R}^n$ is its domain. This abuse in notation is justified if we think of $\mathbb{R}^n$ as the first n coordinates of $\mathbb{R}^{\mathbb{N}}$, and of π_i as the restriction to $\mathbb{R}^n$ of the i -th coordinate projection in $\mathbb{R}^{\mathbb{N}}$.

Buskes, de Pagter and van Rooij showed in [BdPvR91, Theorem 3.7] that every uniformly complete Archimedean vector lattice admits a positively homogeneous continuous function calculus. More precisely, they showed that if X is a uniformly complete Archimedean vector lattice, and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ for some $n \in \mathbb{N}$, then there exists a unique lattice homomorphism $\Phi_{\mathbf{x}}: H_n \rightarrow X$ satisfying $\Phi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$.

What is the natural extension of this function calculus to lattice-ordered algebras? Suppose now that X is an Archimedean uniformly complete lattice-ordered algebra. Let $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ and let $\Phi_{\mathbf{x}}: H_n \rightarrow X$ be the unique lattice homomorphism introduced above. If we want to do function calculus in lattice-ordered algebras, the least we should ask is for the map to be a lattice-algebra homomorphism. But H_n is not an algebra. Still, we can view H_n as a sublattice of $\mathbb{R}^{\mathbb{R}^n}$, the space of all functions $\mathbb{R}^n \rightarrow \mathbb{R}$ equipped with pointwise linear and lattice structures. This space, when also equipped with the pointwise product, is a lattice-ordered algebra, with identity the constant one function $\mathbb{1}$. The weakest extension of the positively homogeneous continuous function calculus to the algebraic setting would be a lattice-algebra homomorphism $\Psi_{\mathbf{x}}: \text{VLA}(H_n) \rightarrow X$, where $\text{VLA}(H_n)$ is the smallest sublattice-algebra of $\mathbb{R}^{\mathbb{R}^n}$ containing H_n . Since X is uniformly complete, we automatically obtain something stronger: $\Psi_{\mathbf{x}}$ extends uniquely to a lattice-algebra homomorphism

on the smallest uniformly complete sublattice-algebra of $\mathbb{R}^{\mathbb{R}^n}$ that contains H_n (this is proved in Lemma 9.7). Thus the question becomes: can we identify the smallest uniformly complete sublattice-algebra of $\mathbb{R}^{\mathbb{R}^n}$ that contains H_n ?

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to have *polynomial growth* if there exist $N \in \mathbb{N}$ and $M \in \mathbb{R}_+$ such that

$$|f| \leq M(1 + |\pi_1| + \cdots + |\pi_n|)^N.$$

The lattice-ordered algebra formed by the continuous polynomial growth functions in $\mathbb{R}^n$ is denoted by PG_n . Consider the elements $f \in PG_n$ for which the limit

$$\lim_{t \rightarrow 0^+} \frac{f(tx)}{t}$$

exists for every $x \in \mathbb{R}^n$, and is uniform on bounded subsets. In this case, we denote $f_h(x) = \lim_{t \rightarrow 0^+} f(tx)t^{-1}$ and say that f_h exists. Observe that $f_h \in H_n$, hence the notation. Denote by PG_n^h the set of elements of PG_n for which f_h exists.

Theorem 9.1 ([BdPvR91, Propositions 3.6 and 4.9]). *Let $n \in \mathbb{N}$.*

- (i) *The uniformly complete vector lattice generated by $\pi_1, \dots, \pi_n$ in $\mathbb{R}^{\mathbb{R}^n}$ is H_n .*
- (ii) *The uniformly complete lattice-ordered algebra generated by $\pi_1, \dots, \pi_n$ in $\mathbb{R}^{\mathbb{R}^n}$ is PG_n^h .*
- (iii) *The uniformly complete lattice-ordered algebra generated by $1, \pi_1, \dots, \pi_n$ in $\mathbb{R}^{\mathbb{R}^n}$ is PG_n .*

We were looking for the smallest uniformly complete lattice-ordered algebra in $\mathbb{R}^{\mathbb{R}^n}$ containing H_n . According to the previous theorem, this is precisely PG_n^h . And the smallest uniformly complete lattice-ordered algebra in $\mathbb{R}^{\mathbb{R}^n}$ containing H_n and 1 is PG_n . In other words, the weakest natural extension of the positively homogeneous continuous function calculus to the algebraic setting is a lattice-algebra homomorphism $\Psi_x^h: PG_n^h \rightarrow X$ (or a lattice-algebra homomorphism $\Psi_x: PG_n \rightarrow X$ satisfying $\Psi_x(1) = 1_X$, in the case that X has an identity 1_X).

Inspired by [LT20], the main goal of this chapter is to study which lattice-ordered algebras with identity admit a polynomial growth continuous function calculus, meaning the following:

Definition 9.2. A lattice-ordered algebra X with identity 1_X admits a *polynomial growth continuous function calculus* if, for each $n \in \mathbb{N}$ and each $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, there is a lattice-algebra homomorphism $\Psi_{\mathbf{x}}: PG_n \rightarrow X$ satisfying

$$\Psi_{\mathbf{x}}(1) = 1_X \text{ and } \Psi_{\mathbf{x}}(\pi_i) = x_i \text{ for } i = 1, \dots, n. \quad (9.1)$$

This is the analogue of [LT20, Definition 1.1], where the authors defined a vector lattice that admits a positively homogeneous continuous function calculus as a vector lattice X for which, for every $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, \dots, x_n) \in X^n$,

there is a vector lattice homomorphism $\Phi_{\mathbf{x}}: H_n \rightarrow X$ satisfying $\Phi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$.

Our focus will be on the case with identity, where we are truly dealing with all functions of polynomial growth. However, the following is a natural definition that will be useful at times.

Definition 9.3. A lattice-ordered algebra X admits a *polynomial growth continuous function calculus without constants* if, for each $n \in \mathbb{N}$ and each $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, there is a lattice-algebra homomorphism $\Psi_{\mathbf{x}}^h: PG_n^h \rightarrow X$ satisfying

$$\Psi_{\mathbf{x}}^h(\pi_i) = x_i \quad \text{for } i = 1, \dots, n. \quad (9.2)$$

The following is a routine, but relevant, observation.

Proposition 9.4. *Let X be a lattice-ordered algebra that admits a polynomial growth continuous function calculus without constants. Then X is a commutative f -algebra.*

Proof. Let X be a lattice-ordered algebra that admits a polynomial growth continuous function calculus without constants. Let $x_1, x_2, x_3 \in X_+$, and suppose $x_1 \wedge x_2 = 0$. Let $\Psi: PG_3^h \rightarrow X$ be a lattice-algebra homomorphism such that $\Psi(\pi_i) = x_i$ for $i = 1, \dots, 3$. Since Ψ is a lattice-algebra homomorphism:

$$\begin{aligned} \Psi(|\pi_1| - |\pi_1| \wedge |\pi_2|) &= |x_1| - |x_1| \wedge |x_2| = x_1, \\ \Psi(|\pi_2| - |\pi_1| \wedge |\pi_2|) &= |x_2| - |x_1| \wedge |x_2| = x_2, \\ \Psi(|\pi_3|) &= |x_3| = x_3. \end{aligned}$$

Since

$$(|\pi_1| - |\pi_1| \wedge |\pi_2|) \wedge (|\pi_2| - |\pi_1| \wedge |\pi_2|) = 0$$

and PG_3^h is an f -algebra, it follows that

$$[|\pi_3|(|\pi_1| - |\pi_1| \wedge |\pi_2|)] \wedge (|\pi_2| - |\pi_1| \wedge |\pi_2|) = 0.$$

Applying Ψ to the previous equality yields $(x_3 x_1) \wedge x_2 = 0$. Using the same idea, it is straightforward to check that X is commutative. Hence X is a commutative f -algebra. $\square$

The previous proposition can be seen as yet another reason why f -algebras are particularly well-behaved and relevant among all lattice-ordered algebras.

To close this circle of ideas, let us note that, just like every Archimedean uniformly complete vector lattice admits a positively homogeneous continuous function calculus, every Archimedean uniformly complete f -algebra with identity admits a polynomial growth continuous function calculus [BdPvR91, Theorem 4.12]. We shall reprove this in the next section using a completely different approach.

9.1.2 Function calculus through free objects

In this section we show how to approach function calculus through free objects. This way of understanding function calculus will be useful in our later proofs, and provides a unified approach to positively homogeneous and polynomial growth (without constants) continuous function calculi. Moreover, it further justifies why polynomial growth continuous function calculus is the natural function calculus to consider on f -algebras.

Recall that a *free vector lattice* over a non-empty set S is a pair $(\text{FVL}(S), \delta)$, where $\text{FVL}(S)$ is a vector lattice and $\delta: S \rightarrow \text{FVL}(S)$ is a map, such that, for every vector lattice X and every map $T: S \rightarrow X$, there exists a unique lattice homomorphism $\hat{T}: \text{FVL}(S) \rightarrow X$ satisfying $\hat{T}\delta = T$. The existence of this object was first established implicitly in [Bir42]; it was later constructed and studied in [Bak68, Ble73].

The free vector lattice over a set S is essentially unique and can be constructed explicitly in the following way: for every $s \in S$, define the point evaluation $\delta_s: \mathbb{R}^S \rightarrow \mathbb{R}$ by $\delta_s(x) = x(s)$; then the free vector lattice is the sublattice generated by $\{\delta_s : s \in S\}$ in $\mathbb{R}^S$ together with the map $\delta: S \rightarrow \text{FVL}(S)$ defined by $\delta(s) = \delta_s$.

Note that, with this particular realization, every element of $\text{FVL}(S)$ is continuous in the product topology; hence $\text{FVL}(S)$ lives inside $C(\mathbb{R}^S)$. Moreover, every element of $\text{FVL}(S)$ is positively homogeneous; hence $\text{FVL}(S)$ lives inside $H(\mathbb{R}^S)$, the vector lattice of positively homogeneous continuous functions defined on $\mathbb{R}^S$. Finally, restricting the functions from $\mathbb{R}^S$ to $\Delta_S = [-1, 1]^S$, $\text{FVL}(S)$ can be regarded as a vector sublattice of $H(\Delta_S)$ (see [dPW15, Lemma 5.1] for the details). Note that the latter space, being a closed sublattice of $C(\Delta_S)$, is a Banach lattice.

One can similarly define the free object in the category of uniformly complete vector lattices. This notion was first considered in [EG24].

Definition 9.5. Let S be a non-empty set. A *free uniformly complete vector lattice* over S is a pair $(\text{FUCVL}(S), \delta)$, where $\text{FUCVL}(S)$ is a uniformly complete vector lattice and $\delta: S \rightarrow \text{FUCVL}(S)$ is a map, such that, for every uniformly complete vector lattice X and every map $T: S \rightarrow X$, there exists a unique lattice homomorphism $\hat{T}: \text{FUCVL}(S) \rightarrow X$ satisfying $T = \hat{T} \circ \delta$.

In [EG24] an abstract construction of this object is provided through the uniform completion (called *r-completion* by the authors): a uniformly complete vector lattice U is a *uniform completion* of a vector lattice X if there exists a vector lattice embedding $i: X \rightarrow U$ such that, for each uniformly complete vector lattice Y and each lattice homomorphism $T: X \rightarrow Y$, there exists a unique lattice homomorphism $S: U \rightarrow Y$ satisfying $T = S \circ i$. Uniform completions are unique up to a unique lattice isomorphism. It is known (see [EG24] for the details) that, if X is a vector sublattice of Y , and Y is uniformly complete, then the uniform completion of X is the intersection of all the uniformly complete vector sublattices of Y containing X .

By checking the universal properties, it is clear that $\text{FUCVL}(S)$ is the uniform completion of $\text{FVL}(S)$. Since $H(\Delta_S)$ is a Banach lattice, it is a uniformly complete vector lattice, and therefore $\text{FUCVL}(S)$ is the intersection of all the uniformly complete vector sublattices of $H(\Delta_S)$ that contain $\text{FVL}(S)$. It follows from Theorem 9.1 that, if S is finite, then $\text{FUCVL}(S) = H(\Delta_S)$, which is in turn isomorphic to $H_{|S|}$. Under this isomorphism, and the identification of S with $\{1, \dots, |S|\}$, the δ_s become the projections π_i . (Note that in this case the free uniformly complete vector lattice coincides with the free Banach lattice over S , see [dPW15, ART18]).

Now we can establish the relation of this theory with positively homogeneous continuous function calculus. As usual, let X be a uniformly complete Archimedean vector lattice, let $n \in \mathbb{N}$ and let $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. The universal property of $\text{FUCVL}(\{1, \dots, n\})$ asserts that there exists a unique lattice homomorphism

$$\text{FUCVL}(\{1, \dots, n\}) \rightarrow X$$

sending δ_i to x_i for $i = 1, \dots, n$. But, as noted above, $\text{FUCVL}(\{1, \dots, n\})$ is essentially H_n and, with this identification, δ_i becomes π_i . Hence the above map is the unique lattice homomorphism $\Phi_{\mathbf{x}}: H_n \rightarrow X$ satisfying $\Phi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$.

So far we have only revisited well-known facts. However, looking back at this approach, there is nothing preventing us from applying it to f -algebras. What kind of function calculus will appear in this case? Of course, it will be the polynomial growth continuous function calculus. The details follow.

A *free Archimedean f -algebra* over a non-empty set S is a pair $(\text{FAfA}(S), \delta)$, where $\text{FAfA}(S)$ is an Archimedean f -algebra and $\delta: S \rightarrow \text{FAfA}(S)$ is a map, such that, for every Archimedean f -algebra A and every map $T: S \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FAfA}(S) \rightarrow A$ satisfying $\hat{T}\delta = T$. Similarly, a *free Archimedean f -algebra with identity* over a non-empty set S is a pair $(\text{FAfA}^1(S), \delta)$, where $\text{FAfA}^1(S)$ is an Archimedean f -algebra with identity and $\delta: S \rightarrow \text{FAfA}^1(S)$ is a map, such that, for every Archimedean f -algebra with identity A and every map $T: S \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FAfA}^1(S) \rightarrow A$ that sends the identity to the identity and satisfies $\hat{T}\delta = T$. The existence of these objects follows from the works [Bir42, HI62].

The free Archimedean f -algebra can be constructed explicitly as the sublattice-algebra generated by $\{\delta_s : s \in S\}$ in $\mathbb{R}^{\mathbb{R}^S}$. Similarly, the free Archimedean f -algebra with identity can be identified with the sublattice-algebra generated by $\{\delta_s : s \in S\} \cup \{\mathbb{1}\}$. Note that, with this particular realization, every element of $\text{FAfA}^1(S)$ is continuous in the product topology; hence $\text{FAfA}(S)$ and $\text{FAfA}^1(S)$ live inside $C(\mathbb{R}^S)$. However, the parallelism with the free vector lattice ends here. The functions of $\text{FAfA}(S)$ cannot be restricted to Δ_S injectively since they are not, in general, positively homogeneous: there exist non-zero elements of $\text{FAfA}(S)$ that vanish on Δ_S .

One can still consider the free object in the category of uniformly complete f -algebras.

Definition 9.6. Let S be a non-empty set. A *free uniformly complete f -algebra* (resp., *free uniformly complete f -algebra with identity*) over S is a pair $(\text{FUCfA}(S), \delta)$ (resp., $(\text{FUCfA}^1(S), \delta)$), where $\text{FUCfA}(S)$ (resp., $\text{FUCfA}^1(S)$) is a uniformly complete f -algebra (resp., f -algebra with identity) and $\delta: S \rightarrow \text{FUCfA}(S)$ (resp., $\delta: S \rightarrow \text{FUCfA}^1(S)$) is a map, such that, for every uniformly complete f -algebra (resp., f -algebra with identity) A and every map $T: S \rightarrow A$, there exists a unique lattice-algebra homomorphism $\hat{T}: \text{FUCfA}(S) \rightarrow A$ satisfying $T = \hat{T} \circ \delta$ (resp., and sending the identity to the identity).

A priori, it is not clear at all whether such an object exists. The following lemma allows for an abstract construction of the free uniformly complete f -algebra as the uniform completion of $\text{FAfA}(S)$.

Lemma 9.7. *Let Y be a uniformly complete f -algebra and let X be a sublattice-algebra of Y . Let $\bar{X}$ be the intersection of all the uniformly complete vector sublattices of Y containing X (i.e., $\bar{X}$ is the uniform completion of X). Then*

- (i) $\bar{X}$ is a sublattice-algebra (and therefore an f -algebra),
- (ii) for every uniformly complete f -algebra A and every lattice-algebra homomorphism $T: X \rightarrow A$ there exists a unique extension of T to a lattice-algebra homomorphism $\bar{T}: \bar{X} \rightarrow A$.

Proof. Item (i) is [BdPvR91, Lemma 2.1]. To show (ii), let A be a uniformly complete f -algebra and let $T: X \rightarrow A$ be a lattice-algebra homomorphism. Since $\bar{X}$ is the uniform completion of X , T extends to a unique lattice homomorphism $\bar{T}: \bar{X} \rightarrow A$. The goal is to show that $\bar{T}$ is an algebra homomorphism. Let $x \in Y_+$. Since Y is an f -algebra, the left multiplication operator $L_x: Y \rightarrow Y$, $L_x(y) = xy$, is a lattice homomorphism. If $x \in X_+$, then we can consider its restriction $L_x|_X: X \rightarrow X$. Since T is an algebra homomorphism, $T \circ L_x|_X = L_{Tx} \circ T$. All the operators in this identity are lattice homomorphisms. Note that $L_x|_{\bar{X}}: \bar{X} \rightarrow \bar{X}$ by item (i), and that $\bar{T} \circ L_x|_{\bar{X}}$ is a lattice homomorphism that coincides with $T \circ L_x|_X$ on X . By uniqueness of the extensions, $\bar{T} \circ L_x|_{\bar{X}}$ is the extension of $T \circ L_x|_X$ to $\bar{X}$. Similarly, $L_{Tx} \circ \bar{T}$ is the extension of $L_{Tx} \circ T$ to $\bar{X}$. From $T \circ L_x|_X = L_{Tx} \circ T$, it follows that $\bar{T} \circ L_x|_{\bar{X}} = L_{Tx} \circ \bar{T}$. Hence, for every $x \in X_+$ and $y \in \bar{X}$,

$$\bar{T}(xy) = (\bar{T} \circ L_x|_{\bar{X}})(y) = (L_{Tx} \circ \bar{T})(y) = Tx \bar{T}y.$$

By linearity, this identity holds for all $x \in X$. Since the product is commutative, we can rewrite the previous equality as $\bar{T} \circ L_y|_X = L_{\bar{T}y} \circ T$, where $L_y|_X: X \rightarrow X$. If $y \in (\bar{X})_+$, then $L_y|_X$ and $L_{\bar{T}y}$ are lattice homomorphisms, and by uniqueness of the extensions $\bar{T} \circ L_y|_{\bar{X}} = L_{\bar{T}y} \circ \bar{T}$. By linearity, this last equation holds for all $y \in \bar{X}$. Hence $\bar{T}$ is an algebra homomorphism. $\square$

The previous lemma implies that $\text{FUCfA}(S)$ (resp. $\text{FUCfA}^1(S)$) does exist: it is the uniform completion of $\text{FAfA}(S)$ (resp. $\text{FAfA}^1(S)$). The following result identifies these free objects when S is finite. It follows immediately from the previous observation and Theorem 9.1.

Theorem 9.8. *Let $n \in \mathbb{N}$ and let $S = \{1, \dots, n\}$.*

- (i) *The free uniformly complete f -algebra over S is PG_n^h together with the map $\delta: S \rightarrow PG_n^h$ given by $\delta(i) = \pi_i$ for $i = 1, \dots, n$.*
- (ii) *The free uniformly complete f -algebra with identity over S is PG_n together with the map $\delta: S \rightarrow PG_n$ given by $\delta(i) = \pi_i$ for $i = 1, \dots, n$.*

It is now time to revisit function calculus through the lens of free objects. Let X be a uniformly complete Archimedean f -algebra, let $n \in \mathbb{N}$ and let $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. The universal property of $\text{FUCfA}(\{1, \dots, n\})$ asserts that there exists a unique lattice-algebra homomorphism

$$\text{FUCfA}(\{1, \dots, n\}) \rightarrow X$$

sending δ_i to x_i for $i = 1, \dots, n$. By the previous theorem, $\text{FUCfA}(\{1, \dots, n\})$ can be identified with PG_n^h , and with this identification δ_i becomes π_i . Hence the above map is the unique lattice-algebra homomorphism $\Psi_{\mathbf{x}}^h: PG_n^h \rightarrow X$ satisfying $\Psi_{\mathbf{x}}^h(\pi_i) = x_i$ for $i = 1, \dots, n$. In the case that X has identity 1_X , this map can be extended to a lattice-algebra homomorphism $\Psi_{\mathbf{x}}: PG_n \rightarrow X$ satisfying also $\Psi_{\mathbf{x}}(1) = 1_X$.

Putting these observations together with Proposition 9.4, we have the following characterization of the Archimedean uniformly complete lattice-ordered algebras that admit a polynomial growth continuous function calculus (without constants).

Proposition 9.9. *Let X be an Archimedean uniformly complete lattice-ordered algebra.*

- (i) *X admits a polynomial growth continuous function calculus without constants if and only if X is an f -algebra.*
- (ii) *X admits a polynomial growth continuous function calculus if and only if X is an f -algebra with identity.*

What about general Archimedean lattice-ordered algebras? Can we characterize those that admit a polynomial growth continuous function calculus? The remainder of this chapter is devoted to obtaining results analogous to those in [LT20] for Archimedean lattice-ordered algebras with identity.

9.2 NECESSARY AND SUFFICIENT CONDITIONS TO ADMIT A POLYNOMIAL GROWTH CONTINUOUS FUNCTION CALCULUS

The goal of this section is to prove the following:

Theorem 9.10. *Let X be an Archimedean lattice-ordered algebra with identity 1_X , and let $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, for some $n \in \mathbb{N}$. The following are equivalent:*

- (i) *There is a lattice-algebra homomorphism $\Psi_{\mathbf{x}}: PG_n \rightarrow X$ with $\Psi_{\mathbf{x}}(1) = 1_X$ and $\Psi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$.*
- (ii) *Let $e = 1_X \vee |x_1| \vee \dots \vee |x_n|$. There exists an f -subalgebra Y of X such that $1_X, x_1, \dots, x_n \in Y$ and, for every $m \in \mathbb{N}$, the norm $\|\cdot\|_{e^m}$ is complete on $Y \cap I_{e^m}$.*
- (iii) *There exist $f \geq 1_X \vee |x_1| \vee \dots \vee |x_n|$ and an f -subalgebra Y of X such that $1_X, x_1, \dots, x_n \in Y$ and, for every $m \in \mathbb{N}$, the norm $\|\cdot\|_{f^m}$ is complete on $Y \cap I_{f^m}$.*

When one and hence all three of these conditions are satisfied, the lattice-algebra homomorphism $\Psi_{\mathbf{x}}$ satisfying the conditions in (i) is unique.

Compare this result with its analogue for positively homogeneous continuous function calculus [LT20, Proposition 2.2]. We can recast the latter result to make the analogy with Theorem 9.10 clearer.

Proposition 9.11. *Let X be an Archimedean vector lattice, and let $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ for some $n \in \mathbb{N}$. The following are equivalent:*

- (i) *There is a vector lattice homomorphism $\Phi_{\mathbf{x}}: H_n \rightarrow X$ with $\Phi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$.*
- (ii) *Let $e = |x_1| \vee \dots \vee |x_n|$. There exists a sublattice Y of X such that $x_1, \dots, x_n \in Y$ and the norm $\|\cdot\|_e$ is complete on $Y \cap I_e$.*
- (iii) *There exists $f \geq |x_1| \vee \dots \vee |x_n|$ and a sublattice Y of X such that $x_1, \dots, x_n \in Y$ and the norm $\|\cdot\|_f$ is complete on $Y \cap I_f$.*

When one and hence all three of these conditions are satisfied, the lattice homomorphism $\Phi_{\mathbf{x}}$ satisfying the conditions in (i) is unique.

Of course, the conditions in Theorem 9.10 are more involved than those in [LT20, Proposition 2.2]. This is not so much due to the fact that we are dealing with the additional algebraic structure (and algebraic homomorphisms), as it is to the fact that the structure of PG_n is more complex than that of H_n . It is well-known that H_n is an AM-space with unit. This allows for much cleaner conditions. Also, the appearance of an f -subalgebra should not surprise the reader in view of Proposition 9.4.

Even though PG_n is not an AM-space with unit, it can be expressed as a countable increasing union of AM-spaces with unit. More precisely, if $d = 1 \vee |\pi_1| \vee \dots \vee |\pi_n|$, then

$$PG_n = \bigcup_{m \in \mathbb{N}} I_{d^m} \quad \text{and} \quad I_{d^m} \subseteq I_{d^{m+1}}.$$

This simple observation is at the core of the proof of Theorem 9.10. Before proceeding to this proof, we need four preliminary lemmas. The first is a slight generalization of [LT20, Lemma 2.1].

Lemma 9.12. *Let X and Y be Archimedean vector lattices, and let $T: X \rightarrow Y$ be a lattice homomorphism. Let $e \in X_+$. Then $T(I_e) \subseteq I_{Te}$, and the restriction $T|_{I_e}: (I_e, \|\cdot\|_e) \rightarrow (I_{Te}, \|\cdot\|_{Te})$ is a contractive operator. Moreover, if $(I_e, \|\cdot\|_e)$ is complete, then the restriction of $\|\cdot\|_{Te}$ to $T(X) \cap I_{Te}$ is complete.*

Proof. If $x \in I_e$, then $|x| \leq \|x\|_e e$ and

$$|Tx| = T|x| \leq \|x\|_e Te$$

because T is a lattice homomorphism. Hence $\|Tx\|_{Te} \leq \|x\|_e$. This proves that $T(I_e) \subseteq I_{Te}$ and that the restriction $T|_{I_e}: (I_e, \|\cdot\|_e) \rightarrow (I_{Te}, \|\cdot\|_{Te})$ is a contractive operator.

Suppose that $(I_e, \|\cdot\|_e)$ is complete. We will show that every absolutely convergent series in $(T(X) \cap I_{Te}, \|\cdot\|_{Te})$ is convergent. Let $x_n \in T(X) \cap I_{Te}$ be a sequence for which the associated series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in $\|\cdot\|_{Te}$. For every $n \in \mathbb{N}$, let $y_n \in X$ be such that $T(y_n) = x_n$, and set $z_n = (y_n \wedge \|x_n\|_{Te} e) \vee (-\|x_n\|_{Te} e)$. Then $|z_n| \leq \|x_n\|_{Te} e$, so $z_n \in I_e$ and $\|z_n\|_e \leq \|x_n\|_{Te}$. Moreover, since $|x_n| \leq \|x_n\|_{Te} Te$ (i.e., $-\|x_n\|_{Te} Te \leq x_n \leq \|x_n\|_{Te} Te$), we have $T(z_n) = (x_n \wedge \|x_n\|_{Te} Te) \vee (-\|x_n\|_{Te} Te) = x_n$. By assumption, $\sum_{n=1}^{\infty} z_n$ converges to a certain $z \in I_e$. Using that $T|_{I_e}$ is linear and continuous, it follows that $\sum_{n=1}^{\infty} x_n = Tz$. $\square$

Lemma 9.13. *Let X be an Archimedean vector lattice. For $n \in \mathbb{N}$, let $I_n \subseteq X$ be a sequence of ideals such that $I_n \subseteq I_{n+1}$ and*

$$X = \bigcup_{n \in \mathbb{N}} I_n.$$

Then X is uniformly complete if and only if I_n is uniformly complete for every $n \in \mathbb{N}$.

Proof. If X is uniformly complete, then every ideal in X is uniformly complete. Conversely, suppose that I_n is uniformly complete for every $n \in \mathbb{N}$. Let $(x_k) \subseteq X$ be a Cauchy sequence in the norm $\|\cdot\|_y$, for some $y \in X_+$. By keeping only a tail of the sequence and relabeling, we may assume that $|x_k - x_l| \leq y$ for all $k, l \in \mathbb{N}$. Since the sequence of ideals I_n is increasing, there exists some $m \in \mathbb{N}$ for which $y, x_1 \in I_m$. Then

$$|x_k| - |x_1| \leq \|x_k - x_1\| \leq |x_k - x_1| \leq y$$

implies that $x_k \in I_m$ for all $k \in \mathbb{N}$. Since I_m is uniformly complete, the sequence (x_k) is uniformly convergent in I_m , and therefore also in X . $\square$

For a positive element e in a lattice-ordered algebra X , denote by IA_e the smallest subspace of X containing e that is, at the same time, an order ideal and a subalgebra.

Lemma 9.14. *Let X be an Archimedean lattice-ordered algebra and let $e \in X_+$ be such that $e^m \in I_{e^{m+1}}$ for all $m \in \mathbb{N}$. Then*

$$IA_e = \bigcup_{m \in \mathbb{N}} I_{e^m}.$$

Proof. Let B be the set on the right hand side. The condition $e^m \in I_{e^{m+1}}$ implies that $I_{e^m} \subseteq I_{e^{m+1}}$ for all $m \in \mathbb{N}$. It follows that B , being a nested union of ideals, is an ideal. To show that it is an algebra, let x and y be elements of B , say $|x| \leq M_1 e^{m_1}$ and $|y| \leq M_2 e^{m_2}$, for some $M_1, M_2 \in \mathbb{R}_+$ and $m_1, m_2 \in \mathbb{N}$. Then

$$|xy| \leq |x||y| \leq M_1 M_2 e^{m_1+m_2}$$

and therefore $xy \in I_{e^{m_1+m_2}} \subseteq B$. Hence B is a subalgebra and an order ideal that contains e . In particular, $IA_e \subseteq B$. Conversely, since IA_e is a subalgebra containing e , $e^m \in IA_e$ and, since it is an order ideal, $I_{e^m} \subseteq IA_e$ for every $m \in \mathbb{N}$. It follows that $B \subseteq IA_e$. $\square$

Remark 9.15. If X has identity 1_X , then the previous lemma applies to every $e \geq 1_X$.

The final preliminary lemma is only needed for uniqueness.

Lemma 9.16. *Let X and Y be Archimedean vector lattices, with X uniformly complete, and let $T: X \rightarrow Y$ be a positive map. Let $X' \subseteq X$ and $Y' \subseteq Y$ be sublattices, with Y' uniformly complete. Denote by $\overline{X'}$ the uniform closure of X' in X . If $T(X') \subseteq Y'$, then also $T(\overline{X'}) \subseteq Y'$.*

Proof. To prove this result, we need a different description of the uniform closure. Define a transfinite sequence $(Z_\beta)_{\beta < \omega_1}$ of sublattices of X as follows: $Z_1 = X'$,

$$Z_{\beta+1} = \{x \in X : x_n \xrightarrow{u} x(y), \text{ for some } x_n, y \in Z_\beta\}$$

for all $\beta < \omega_1$ and, if $\gamma < \omega_1$ is a limit ordinal, then $Z_\gamma = \bigcup_{\beta < \gamma} Z_\beta$. It is then true that

$$\overline{X'} = \bigcup_{\beta < \omega_1} Z_\beta \tag{9.3}$$

(see [EG24, Lemma 4]). We will prove by transfinite induction on β that $T(Z_\beta) \subseteq Y'$.

To do so, first note that $x_n \xrightarrow{u} x(y)$ in X implies $Tx_n \xrightarrow{u} Tx(Ty)$ in Y by the positivity of T . By assumption, $T(Z_1) \subseteq Y'$. Suppose that $T(Z_\beta) \subseteq Y'$ and let $x \in Z_{\beta+1}$. Then there exist $x_n, y \in Z_\beta$ such that $x_n \xrightarrow{u} x(y)$. Applying T it follows that $Tx_n \xrightarrow{u} Tx(Ty)$ with $Tx_n, Ty \in Y'$. Since Y' is uniformly complete, and uniform limits are unique, we must have $Tx \in Y'$. Hence $T(Z_{\beta+1}) \subseteq Y'$. Finally, if $\gamma < \omega_1$ is a limit ordinal and $T(Z_\beta) \subseteq Y'$ for all $\beta < \gamma$, then

$$T(Z_\gamma) = T\left(\bigcup_{\beta < \gamma} Z_\beta\right) = \bigcup_{\beta < \gamma} T(Z_\beta) \subseteq Y'.$$

From (9.3) it follows that $T(\overline{X'}) \subseteq Y'$. $\square$

Proof of Theorem 9.10. First we show that (i) implies (ii). Let $\Psi_x: PG_n \rightarrow X$ be a lattice-algebra homomorphism with $\Psi_x(1) = 1_X$ and $\Psi_x(\pi_i) = x_i$ for $i = 1, \dots, n$. Put $Y = \Psi_x(PG_n)$. Since PG_n is an f -algebra, and Ψ_x is a

lattice-algebra homomorphism, it follows that Y is an f -subalgebra of X (see [Bou02, Proposition 3.2]). Denote $d = 1 \vee |\pi_1| \vee \cdots \vee |\pi_n|$. It is clear that $\Psi_{\mathbf{x}}(d^m) = e^m$. Also, since PG_n is uniformly complete, $(I_{d^m}, \|\cdot\|_{d^m})$ is complete. By Lemma 9.12, $\|\cdot\|_{e^m}$ is complete on $Y \cap I_{e^m}$.

That (ii) implies (iii) is obvious. To show that (iii) implies (i), let $f \in X_+$ be such that $f \geq 1_X \vee |x_1| \vee \cdots \vee |x_n|$, and let Y be an f -subalgebra of X satisfying the conditions of (iii). Put $Y' = IA_f \cap Y$. Note that $1_X, x_1, \dots, x_n \in Y'$. Being the intersection of sublattice-algebras, Y' is again a sublattice-algebra of X . In fact, since $Y' \subseteq Y$, Y' is also an f -subalgebra of X . By assumption, $f \geq 1_X$, and by Lemma 9.14,

$$IA_f = \bigcup_{m \in \mathbb{N}} I_{f^m}.$$

Therefore

$$Y' = Y' \cap IA_f = \bigcup_{m \in \mathbb{N}} Y' \cap I_{f^m}.$$

Note that, for every $m \in \mathbb{N}$, $Y' \cap I_{f^m}$ is an ideal in Y' and $Y' \cap I_{f^m} \subseteq Y' \cap I_{f^{m+1}}$. By assumption, $\|\cdot\|_{f^m}$ is complete on $Y \cap I_{f^m} = Y' \cap I_{f^m}$; being a Banach lattice, $Y' \cap I_{f^m}$ is uniformly complete. By Lemma 9.13, Y' is uniformly complete. In conclusion, Y' is a uniformly complete f -algebra with identity 1_X that contains $x_1, \dots, x_n$. By Proposition 9.9, there exists a lattice-algebra homomorphism $\Psi_{\mathbf{x}}: PG_n \rightarrow Y'$ satisfying $\Psi_{\mathbf{x}}(1) = 1_X$ and $\Psi_{\mathbf{x}}(\pi_i) = x_i$ for $i = 1, \dots, n$. The composition of this map with the natural inclusion $Y' \subseteq X$ proves that (i) holds.

It only remains to prove uniqueness. Suppose that a map $\Psi_{\mathbf{x}}$ such as that in (i) exists. We have proved above that there exists a uniformly complete f -algebra Y' with identity 1_X such that $x_1, \dots, x_n \in Y'$. Note that $\Psi_{\mathbf{x}}(\text{VLA}\{1, \pi_1, \dots, \pi_n\}) \subseteq Y'$. According to Theorem 9.1, PG_n is the uniform closure of $\text{VLA}\{1, \pi_1, \dots, \pi_n\}$. Applying Lemma 9.16 it follows that $\Psi_{\mathbf{x}}(PG_n) \subseteq Y'$. Recall from Theorem 9.8 that PG_n is the free object in the category of uniformly complete f -algebras with identity. Hence $\Psi_{\mathbf{x}}$ must be the composition of the unique map $PG_n \rightarrow Y'$ sending 1 to 1_X and π_i to x_i , for $i = 1, \dots, n$, with the natural inclusion $Y' \subseteq X$. $\square$

As a consequence of this theorem, we obtain a characterization of the lattice-ordered algebras with identity that admit a polynomial growth continuous function calculus.

Corollary 9.17. *Let X be an Archimedean lattice-ordered algebra with identity 1_X . Then X admits a polynomial growth continuous function calculus if and only if X is an f -algebra and, for every $x_1, \dots, x_n \in X$, there exists $f \geq 1_X \vee |x_1| \vee \cdots \vee |x_n|$ such that, for every $m \in \mathbb{N}$, the norm $\|\cdot\|_{f^m}$ is complete on the closed sublattice generated by $x_1, \dots, x_n$ in I_{f^m} .*

In this case, the polynomial growth continuous function calculus is unique (in the sense that for each $n \in \mathbb{N}$ and $\mathbf{x} \in X^n$ there is only one lattice-algebra homomorphism $\Psi_{\mathbf{x}}: PG_n \rightarrow X$ which satisfies (9.1)), and

$$\Psi_{(\Psi_{\mathbf{x}}(f_1), \dots, \Psi_{\mathbf{x}}(f_m))}(g) = \Psi_{\mathbf{x}}(g \circ (f_1 \times \cdots \times f_m)) \quad (9.4)$$

for each $m, n \in \mathbb{N}$, $\mathbf{x} \in X^n$, $f_1, \dots, f_m \in PG_n$ and $g \in PG_m$, where $f_1 \times \dots \times f_m: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the function defined by

$$(f_1 \times \dots \times f_m)(t) = (f_1(t), \dots, f_m(t)), \quad t \in \mathbb{R}^n.$$

Proof. If X admits a polynomial growth continuous function calculus, then X is an f -algebra by Proposition 9.4. And if $x_1, \dots, x_n \in X$, then $e = 1_X \vee |x_1| \vee \dots \vee |x_n|$ has the stated properties by Theorem 9.10. The converse also follows directly from Theorem 9.10.

Uniqueness was proved in Theorem 9.10. To show (9.4), let $f_1, \dots, f_m \in PG_n$ and $g \in PG_m$ be as stated. First we need to check that $g \circ (f_1 \times \dots \times f_m) \in PG_n$. Suppose $|g| \leq M(1 + |\pi_1| + \dots + |\pi_m|)^N$ for some $M \in \mathbb{R}_+$ and $N \in \mathbb{N}$. Then it is immediate to check that

$$|g \circ (f_1 \times \dots \times f_m)| \leq M(1 + |f_1| + \dots + |f_m|)^N.$$

Since $f_1, \dots, f_m \in PG_n$, the right hand side of this equation belongs to PG_n . Since PG_n is an ideal in $C(\mathbb{R}^n)$, and $g \circ (f_1 \times \dots \times f_m)$ is continuous, it follows that $g \circ (f_1 \times \dots \times f_m) \in PG_n$. Hence, it makes sense to evaluate $\Psi_{\mathbf{x}}$ at $g \circ (f_1 \times \dots \times f_m)$. This is true of every $g \in PG_m$. It is immediate to check that the map

$$\begin{array}{ccc} PG_m & \longrightarrow & X \\ g & \longmapsto & \Psi_{\mathbf{x}}(g \circ (f_1 \times \dots \times f_m)) \end{array}$$

is a lattice-algebra homomorphism that maps π_i to $\Psi_{\mathbf{x}}(f_i)$ for $i = 1, \dots, m$. By uniqueness, it must be the map $\Psi_{(\Psi_{\mathbf{x}}(f_1), \dots, \Psi_{\mathbf{x}}(f_m))}$. $\square$

To complete this section, we present three examples illustrating Theorem 9.10. The first one shows that, even if X is a uniformly complete lattice-ordered algebra with identity, it may not contain sufficiently many f -subalgebras to admit a polynomial growth continuous function calculus.

Example 9.18. Consider $\mathbb{R}^2$ with the usual vector lattice structure. Define the product

$$(x, y)(x', y') = (xx' + xy' + yx', yy').$$

It is immediate to check that $\mathbb{R}^2$ with this product is a lattice-ordered algebra with identity $(0, 1)$. But it is not an f -algebra, because $(1, 0)(0, 1) = (1, 0)$. The element $(1, 0)$ is such that, if Y is a sublattice-algebra containing both $(1, 0)$ and the identity $(0, 1)$, then $Y = \mathbb{R}^2$, which is not an f -algebra. Hence there is no f -subalgebra containing $(1, 0)$ and the identity. By Theorem 9.10, we cannot perform polynomial growth continuous function calculus on $(1, 0)$.

The second example fails the other condition: it is an f -algebra, but it does not contain sublattice-algebras Y for which the norm $\|\cdot\|_{e^m}$ is complete on $Y \cap I_{e^m}$ (where we are using the notation of Theorem 9.10).

Example 9.19. Consider the function $I: [0, 1] \rightarrow [0, 1]$ given by $I(x) = x$ for all $x \in [0, 1]$. Let X be the sublattice-algebra of $C[0, 1]$ generated by I and $\mathbb{1}_{[0, 1]}$.

Note that X is an f -algebra with identity. We have $\mathbb{1}_{[0,1]} \vee I = \mathbb{1}_{[0,1]}$ and the ideal generated by $\mathbb{1}_{[0,1]}$ in X is X . Yet X is not complete for $\|\cdot\|_{\mathbb{1}_{[0,1]}}$. Indeed, this norm is precisely the uniform norm, and X is not closed in the uniform norm by the Stone–Weierstrass theorem.

The final example is a positive one: it shows how Corollary 9.17 can be used in practice to prove that a non-uniformly complete lattice-ordered algebra with identity admits a polynomial growth continuous function calculus. This example is a continuation of [LT20, Proposition 3.1].

Example 9.20. Let K be a compact Hausdorff space of infinite cardinality, and let $s_0 \in K$ be an accumulation point of a countably infinite subset of K . (For instance, $K = [0, 1]$ and $s_0 = 0$.) Let

$$X = \{f \in C(K) : f \text{ is constant on some neighborhood of } s_0\}.$$

It was shown in [LT20, Proposition 3.1] that this set is a proper and dense sublattice of $C(K)$ that is not uniformly complete. It is easy to check that it is a subalgebra and that $\mathbb{1}_K \in X$. Therefore, X is an Archimedean f -algebra with identity. We will show that it admits a polynomial growth continuous function calculus by checking the condition of Corollary 9.17.

Let $f_1, \dots, f_n \in X$ and let U be a neighborhood of s_0 in which $f_1|_U, \dots, f_n|_U$ are all constant. Let $c = \max\{\|f_1\|_\infty, \dots, \|f_n\|_\infty, 1\}$, and let

$$Y = \{f \in C(K) : f|_U \text{ is constant}\}.$$

Then $e = c\mathbb{1}_K$ satisfies $e \geq \mathbb{1}_K \vee |f_1| \vee \dots \vee |f_n|$ and $f_1, \dots, f_n \in Y$. Note that Y is a closed sublattice-algebra of $C(K)$. Fixed $m \in \mathbb{N}$, $I_{e^m} = X$ and $\|\cdot\|_{e^m}$ is the uniform norm; it follows directly that $\|\cdot\|_{e^m}$ is complete on $I_{e^m} \cap Y$. Hence, even though X is not uniformly complete, it admits a polynomial growth continuous function calculus.

9.3 VECTOR SPACES WITH POLYNOMIAL GROWTH CONTINUOUS FUNCTION CALCULUS

Our final result is an extension of [LT20, Theorem 1.4]. It states that if a vector space admits a sufficiently strong polynomial growth continuous function calculus without constants, then it is automatically a commutative f -algebra.

Theorem 9.21. *Let X be a vector space and suppose that, for each $n \in \mathbb{N}$ and each $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, there is a linear map $\Psi_{\mathbf{x}} : PG_n^h \rightarrow X$ which satisfies (9.2) and (9.4). Then X can be endowed with the structure of a commutative f -algebra that admits a polynomial growth continuous function calculus without constants.*

Proof. For $n \in \mathbb{N}$ and $f \in PG_n^h$, denote $\Psi_{\mathbf{x}}(f)$ by $f(\mathbf{x})$. In this notation, the linearity of $\Psi_{\mathbf{x}}$ reads

$$(f + \lambda g)(\mathbf{x}) = f(\mathbf{x}) + \lambda g(\mathbf{x})$$

for all $f, g \in PG_n^h$ and $\lambda \in \mathbb{R}$. Condition (9.2) becomes $\pi_i(\mathbf{x}) = x_i$, while (9.4) has the form

$$g(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) = (g \circ (f_1 \times \dots \times f_m))(\mathbf{x}).$$

Consider the function of PG_2 :

$$\begin{aligned} \sigma: \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (t_1, t_2) &\longmapsto t_1 \vee t_2 \end{aligned}$$

and define a partial order $\leq$ on X by setting $x_1 \leq x_2$ if and only if $\sigma(x_1, x_2) = x_2$. It is shown in [LT20, Theorem 1.4] that this order makes X into a vector lattice. Moreover, the supremum of $x_1, x_2 \in X$ in this order is given by $\sigma(x_1, x_2)$.

Consider the function of PG_2 :

$$\begin{aligned} \rho: \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (t_1, t_2) &\longmapsto t_1 t_2. \end{aligned}$$

Define a product on X by

$$\begin{aligned} X \times X &\longrightarrow X \\ (x_1, x_2) &\longmapsto \rho(x_1, x_2). \end{aligned}$$

We claim that X , with this product, is a commutative real algebra. Let us check the required axioms.

COMMUTATIVITY. Since the product in $\mathbb{R}$ is commutative, $\rho(t_1, t_2) = \rho(t_2, t_1)$ holds for all $t_1, t_2 \in \mathbb{R}$. In other words, the identity $\rho \circ \pi_1 \times \pi_2 = \rho \circ \pi_2 \times \pi_1$ holds in PG_2 . Using (9.2) and (9.4), the following holds for all $x_1, x_2 \in X$:

$$(\rho \circ \pi_1 \times \pi_2)(x_1, x_2) = \rho(\pi_1(x_1, x_2), \pi_2(x_1, x_2)) = \rho(x_1, x_2).$$

Similarly, $(\rho \circ \pi_2 \times \pi_1)(x_1, x_2) = \rho(x_2, x_1)$. Hence $\rho(x_1, x_2) = \rho(x_2, x_1)$.

ASSOCIATIVITY. The associativity of $\mathbb{R}$ means that $\rho(\rho(t_1, t_2), t_3) = \rho(t_1, \rho(t_2, t_3))$ for all $t_1, t_2, t_3 \in \mathbb{R}$. This, in turn, implies that the following identity holds in PG_3 :

$$\rho \circ [(\rho \circ \pi_1 \times \pi_2) \times \pi_3] = \rho \circ [\pi_1 \times (\rho \circ \pi_2 \times \pi_3)].$$

Applying this to $(x_1, x_2, x_3) \in X^3$ and simplifying with (9.2) and (9.4) yields $\rho(\rho(x_1, x_2), x_3) = \rho(x_1, \rho(x_2, x_3))$.

DISTRIBUTIVITY. Distributivity in $\mathbb{R}$ translates into the identity

$$\rho \circ [\pi_1 \times (\pi_2 + \pi_3)] = \rho \circ \pi_1 \times \pi_2 + \rho \circ \pi_1 \times \pi_3.$$

Applying this to $(x_1, x_2, x_3) \in X^3$, using linearity and applying (9.2) and (9.4) yields $\rho(x_1, x_2 + x_3) = \rho(x_1, x_2) + \rho(x_1, x_3)$. Distributivity on the right follows from commutativity and distributivity on the left.

ASSOCIATIVITY WITH SCALARS. Fixed $\lambda, \mu \in \mathbb{R}$, associativity and commutativity in $\mathbb{R}$ imply $\rho(\lambda t_1, \mu t_2) = \lambda \mu \rho(t_1, t_2)$ for all $t_1, t_2 \in \mathbb{R}$. This means that

$$\rho \circ (\lambda \pi_1) \times (\mu \pi_2) = (\lambda \mu)(\rho \circ \pi_1 \times \pi_2).$$

Evaluating at $(x_1, x_2) \in X^2$ and using linearity and (9.2) and (9.4) gives $\rho(\lambda x_1, \mu x_2) = \lambda \mu \rho(x_1, x_2)$.

Hence X is a real algebra when equipped with the product $x_1 x_2 = \rho(x_1, x_2)$. Next we show that it is also a lattice-ordered algebra. For this we have to check that, if $x_1, x_2 \in X_+$, then $x_1 x_2 \in X_+$. Since in $\mathbb{R}$ the product of positive elements is positive, the identity

$$[(t_1 \vee 0)(t_2 \vee 0)] \vee 0 = (t_1 \vee 0)(t_2 \vee 0)$$

holds for all $t_1, t_2 \in \mathbb{R}$. This implies that

$$\sigma \circ [\rho \circ (\sigma \circ \pi_1 \times 0) \times (\sigma \circ \pi_2 \times 0)] = \rho \circ [(\sigma \circ \pi_1 \times 0) \times (\sigma \circ \pi_2 \times 0)],$$

where $0: \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes the constant zero function. Linearity of Ψ_x implies that $0(x_1, x_2) = 0$ for all $(x_1, x_2) \in X^2$. Therefore the evaluation of the previous identity at (x_1, x_2) yields

$$\sigma[\rho(\sigma(x_1, 0), \sigma(x_2, 0)), 0] = \rho[\sigma(x_1, 0), \sigma(x_2, 0)].$$

This equation holds for all $x_1, x_2 \in X$. When $x_1, x_2 \geq 0$ (that is, when $\sigma(x_i, 0) = x_i$ for $i = 1, 2$) this equation becomes

$$\sigma[\rho(x_1, x_2), 0] = \rho(x_1, x_2).$$

This is, by definition, the same as $x_1 x_2 = \rho(x_1, x_2) \geq 0$. Hence X is a lattice-ordered algebra.

To show that X is an f -algebra, we will use the following fact due to G. Birkhoff [Bir61, Lemma 3]: the lattice-ordered algebra X is an f -algebra if and only if the equalities

$$(x_1 \vee 0) \wedge [((-x_1) \vee 0)(x_2 \vee 0)] = 0 = (x_1 \vee 0) \wedge [(x_2 \vee 0)((-x_1) \vee 0)]$$

hold for all $(x_1, x_2) \in X^2$. Since X is commutative, we need only check the identity on the left. Since $\mathbb{R}$ is an f -algebra,

$$(t_1 \vee 0) \wedge [((-t_1) \vee 0)(t_2 \vee 0)] = 0$$

holds for all $t_1, t_2 \in \mathbb{R}$; let us rewrite it as

$$(-(t_1 \vee 0)) \vee [-((-t_1) \vee 0)(t_2 \vee 0)] = 0$$

so that only suprema appear in the formula. This implies that

$$\sigma \circ [-\sigma \circ (\pi_1 \times 0)] \times [-\rho \circ ((\sigma \circ ((-\pi_1) \times 0)) \times (\sigma \circ (\pi_2 \times 0)))] = 0.$$

Applying this identity to $(x_1, x_2) \in X^2$ yields

$$\sigma(-\sigma(x_1, 0), -\rho(\sigma(-x_1, 0), \sigma(x_2, 0))) = 0.$$

By [Bir61, Lemma 3], X is a commutative f -algebra.

Finally, to show that X admits a polynomial growth continuous function calculus without constants, it suffices to check that $\Psi_{\mathbf{x}}: PG_n^h \rightarrow X$ is a lattice-algebra homomorphism for all $\mathbf{x} = (x_1, \dots, x_n) \in X^n$. To make the equations clearer, denote $\sigma(x_1, x_2)$ by $x_1 \vee x_2$ and $\rho(x_1, x_2)$ by $x_1 x_2$. Then for all $f, g \in PG_n^h$:

$$\Psi_{\mathbf{x}}(f \vee g) = \Psi_{\mathbf{x}}(\sigma \circ (f \times g)) = (\sigma \circ (f \times g))(\mathbf{x}) = \sigma(f(\mathbf{x}), g(\mathbf{x})) = \Psi_{\mathbf{x}}(f) \vee \Psi_{\mathbf{x}}(g),$$

and also

$$\Psi_{\mathbf{x}}(fg) = \Psi_{\mathbf{x}}(\rho \circ (f \times g)) = (\rho \circ (f \times g))(\mathbf{x}) = \rho(f(\mathbf{x}), g(\mathbf{x})) = \Psi_{\mathbf{x}}(f) \Psi_{\mathbf{x}}(g). \quad \square$$

CONCLUSIONS

This thesis develops the structure and representation theory of Banach lattice algebras and Banach f -algebras, largely organised around the cluster of problems raised by Wickstead in 2017 [Wic17b, Wic17a]. The main contributions are summarised below.

The principal outcome is a negative answer to Wickstead's representation question: there exists a two-dimensional Banach lattice algebra with identity that admits no faithful representation as regular operators on any order complete Banach lattice (Example 4.22). The argument also proves Wickstead's conjecture on positive projections with constant diagonal (Corollary 4.15).

To study the interplay between order and algebra in $\mathcal{L}_r(X)$, we introduced the class of inner band projections (Theorem 2.1) and proved that every band projection on $\mathcal{L}_r(X)$ is inner if and only if X is atomic and order continuous (Theorem 2.19).

We characterized the Banach lattice algebras that are isometric to a closed sublattice-algebra of $C(K)$ as those admitting a positive net which is simultaneously an approximate order unit and an approximate algebraic identity (Theorem 6.15).

We classified all f -algebra products on AM-spaces and on AL-spaces. For an AM-space X , the f -algebra products are in one-to-one correspondence with the positive cone of an associated AM-space W_X (Theorem 7.4). For an AL-space, they are parametrized by the positive bounded weights indexed by a maximal family of norm-one atoms (Theorem 7.17).

We constructed the free Banach f -algebra $\text{FB}fA[E]$ generated by a Banach space E . As a key ingredient we proved a structure theorem for Banach f -algebras: every Banach f -algebra admits an injective and contractive lattice-algebra homomorphism into $X_0 \oplus_\infty C(K)$, where X_0 is a Banach lattice equipped with the zero product (Theorem 8.20). Combined with an explicit description of the kernel, this yielded an injective contractive lattice-algebra homomorphism $\iota: \text{FN}fA[E] \rightarrow C(B_{E^*})$ representing the free normed f -algebra as continuous functions on the unit ball of E^* (Corollary 8.51). When E is finite-dimensional, we showed that $\text{FB}fA[E]$ is lattice-algebra isomorphic to $C([0, 1] \times S_{E^*})$ equipped with pointwise order and the weighted product $(f \star g)(r, u) = rf(r, u)g(r, u)$ (Theorem 8.56).

Finally, we characterized the Archimedean lattice-ordered algebras with identity that admit a polynomial growth continuous function calculus (Theorem 9.10 and Corollary 9.17).

CONCLUSIONES

Esta tesis desarrolla la estructura y la teoría de representación de las álgebras-retículos de Banach y de las f -álgebras de Banach, organizada en gran medida en torno al conjunto de problemas planteados por Wickstead en 2017 [Wic17b, Wic17a]. A continuación se resumen las principales aportaciones.

El resultado principal es una respuesta negativa a la pregunta de representación de Wickstead: existe un álgebra-retículo de Banach bidimensional con identidad que no admite ninguna representación fiel como operadores regulares sobre ningún retículo de Banach orden completo (Example 4.22). El argumento también demuestra la conjetura de Wickstead sobre las proyecciones positivas con diagonal constante (Corollary 4.15).

Para estudiar la interacción entre las estructuras de orden y algebraica en $\mathcal{L}_r(X)$, introdujimos la clase de proyecciones en banda internas (Theorem 2.1) y demostramos que toda proyección en banda sobre $\mathcal{L}_r(X)$ es interna si y solo si X es atómico y orden continuo (Theorem 2.19).

Caracterizamos también las álgebras-retículos de Banach que pueden ser embebidas isométricamente como subálgebra-retículo cerrado de $C(K)$: son precisamente aquellas que admiten una red positiva que es simultáneamente una unidad de orden aproximada y una unidad algebraica aproximada (Theorem 6.15).

Clasificamos todos los productos de f -álgebra sobre los AM-espacios y sobre los AL-espacios. Para un AM-espacio X , los productos de f -álgebra están en correspondencia biunívoca con el cono positivo de un AM-espacio asociado W_X (Theorem 7.4). Para un AL-espacio, están parametrizados por los pesos positivos acotados indexados por una familia maximal de átomos de norma uno (Theorem 7.17).

Construimos la f -álgebra de Banach libre $\text{FB}fA[E]$ generada por un espacio de Banach E . Como ingrediente clave, demostramos un teorema de estructura para las f -álgebras de Banach: toda f -álgebra de Banach admite un homomorfismo álgebra-retículo inyectivo y contractivo en $X_0 \oplus_\infty C(K)$, donde X_0 es un retículo de Banach equipado con el producto cero (Theorem 8.20). Combinado con una descripción explícita del núcleo, esto proporcionó un homomorfismo álgebra-retículo inyectivo y contractivo $\iota: \text{FN}fA[E] \rightarrow C(B_{E^*})$ que representa la f -álgebra normada libre como funciones continuas sobre la bola unidad de E^* (Corollary 8.51). Cuando E es de dimensión finita, demostramos que $\text{FB}fA[E]$ es isomorfa como álgebra-retículo a $C([0, 1] \times S_{E^*})$ equipado con el orden puntual y el producto $(f \star g)(r, u) = rf(r, u)g(r, u)$ (Theorem 8.56).

Finalmente, caracterizamos las álgebras reticuladas arquimedianas con identidad que admiten un cálculo funcional continuo de crecimiento polinómico (Theorem 9.10 and Corollary 9.17).

BIBLIOGRAPHY

- [AA02a] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002. MR [1921782](#)
- [AA02b] Y. A. Abramovich and C. D. Aliprantis, *Problems in operator theory*, Graduate Studies in Mathematics, vol. 51, American Mathematical Society, Providence, RI, 2002. MR [1921783](#)
- [AB06] Charalambos D. Aliprantis and Owen Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006, Reprint of the 1985 original. MR [2262133](#)
- [AD25] Youssef Azouzi and Wassim Dhifaoui, *Order denseness in free Banach lattices*, Preprint, arXiv:2508.11648 [math.FA], 2025.
- [Ale12] Egor A. Alekhno, *The irreducibility in ordered Banach algebras*, Positivity **16** (2012), no. 1, 143–176. MR [2892579](#)
- [Ale18] Egor A. Alekhno, *On the peripheral spectrum of positive elements*, Positivity **22** (2018), no. 4, 931–968. MR [3843450](#)
- [AMCRA20] Antonio Avilés, Gonzalo Martínez-Cervantes, and José David Rodríguez Abellán, *On projective Banach lattices of the form $C(K)$ and $FBL[E]$* , J. Math. Anal. Appl. **489** (2020), no. 1, 124129, 11. MR [4083125](#)
- [AMCRAZ22] Antonio Avilés, Gonzalo Martínez-Cervantes, José David Rodríguez Abellán, and Abraham Rueda Zoca, *Lattice embeddings in free Banach lattices over lattices*, Math. Inequal. Appl. **25** (2022), no. 2, 495–509. MR [4428584](#)
- [AMCRZT22] Antonio Avilés, Gonzalo Martínez-Cervantes, Abraham Rueda Zoca, and Pedro Tradacete, *Linear versus lattice embeddings between Banach lattices*, Adv. Math. **406** (2022), 14. MR [4452664](#)
- [APRA18] Antonio Avilés, Grzegorz Plebanek, and José David Rodríguez Abellán, *Chain conditions in free Banach lattices*, J. Math. Anal. Appl. **465** (2018), no. 2, 1223–1229. MR [3809354](#)

- [ARA19] Antonio Avilés and José David Rodríguez Abellán, *The free Banach lattice generated by a lattice*, Positivity **23** (2019), no. 3, 581–597. MR [3977585](#)
- [ART18] Antonio Avilés, José Rodríguez, and Pedro Tradacete, *The free Banach lattice generated by a Banach space*, J. Funct. Anal. **274** (2018), no. 10, 2955–2977. MR [3777636](#)
- [AT23] Antonio Avilés and Pedro Tradacete, *Amalgamation and injectivity in banach lattices*, Int. Math. Res. Not. **2023** (2023), no. 2, 956–997. MR [4537317](#)
- [ATV19] Antonio Avilés, Pedro Tradacete, and Ignacio Villanueva, *The free Banach lattices generated by ℓ_p and c_0* , Rev. Mat. Complut. **32** (2019), no. 2, 353–364. MR [3942920](#)
- [AVK79] Ju.Ā. Abramovič, A. I. Veksler, and A. V. Koldunov, *Operators that preserve disjunction*, Dokl. Akad. Nauk SSSR **248** (1979), no. 5, 1033–1036. MR [553919](#)
- [Bak68] Kirby A. Baker, *Free vector lattices*, Canadian J. Math. **20** (1968), 58–66. MR [224524](#)
- [BB06] Karim Boulabiar and Gerard Buskes, *Vector lattice powers: f -algebras and functional calculus*, Comm. Algebra **34** (2006), no. 4, 1435–1442. MR [2224884](#)
- [BBT07] K. Boulabiar, G. Buskes, and A. Triki, *Results in f -algebras*, Positivity, Trends Math., Birkhäuser, Basel, 2007, pp. 73–96. MR [2382215](#)
- [BD73] Frank F. Bonsall and John Duncan, *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], Band 80, Springer-Verlag, New York-Heidelberg, 1973. MR [423029](#)
- [BdPvR91] G. Buskes, B. de Pagter, and A. van Rooij, *Functional calculus on Riesz spaces*, Indag. Math. (N.S.) **2** (1991), no. 4, 423–436. MR [1149692](#)
- [BGSdH⁺26] Eugene Bilokopytov, Enrique García-Sánchez, David de Hevia, Gonzalo Martínez-Cervantes, and Pedro Tradacete, *Norm-attaining lattice homomorphisms and renormings of Banach lattices*, J. Funct. Anal. **290** (2026), no. 4. MR [4983374](#)
- [BH90a] S. J. Bernau and C. B. Huijsmans, *Almost f -algebras and d -algebras*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 287–308. MR [1027782](#)

-
- [BH90b] S. J. Bernau and C. B. Huijsmans, *On the positivity of the unit element in a normed lattice ordered algebra*, Studia Math. **97** (1990), no. 2, 143–149. MR [1083344](#)
- [BH95] S. J. Bernau and C. B. Huijsmans, *The order bidual of almost f -algebras and d -algebras*, Trans. Amer. Math. Soc. **347** (1995), no. 11, 4259–4275. MR [1308002](#)
- [Bir35] Garrett Birkhoff, *On the structure of abstract algebras*, Proc. Camb. Philos. Soc. **31** (1935), 433–454.
- [Bir42] Garrett Birkhoff, *Lattice-ordered groups*, Ann. of Math. (2) **43** (1942), 298–331. MR [6550](#)
- [Bir61] Garrett Birkhoff, *Lattice theory*, American Mathematical Society Colloquium Publications, vol. XXV, American Mathematical Society, Providence, RI, 1961, Revised ed. MR [123490](#)
- [BK69] Alain Bigard and Klaus Keimel, *Sur les endomorphismes conservant les polaires d'un groupe réticulé archimédien*, Bull. Soc. Math. France **97** (1969), 381–398. MR [262137](#)
- [BKW77] Alain Bigard, Klaus Keimel, and Samuel Wolfenstein, *Groupes et anneaux réticulés*, Lecture Notes in Mathematics, vol. 608, Springer-Verlag, Berlin-New York, 1977. MR [552653](#)
- [Bla06] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^* -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. MR [2188261](#)
- [Bla25] A. Blanco, *Some results and questions on r -amenability*, Positivity **29** (2025), no. 1, Paper No. 7, 54. MR [4836153](#)
- [Ble73] Roger D. Bleier, *Free vector lattices*, Trans. Amer. Math. Soc. **176** (1973), 73–87. MR [311541](#)
- [Bou02] Karim Boulabiar, *The range of lattice homomorphisms on f -algebras*, Ordered algebraic structures, Dev. Math., vol. 7, Kluwer Acad. Publ., Dordrecht, 2002, pp. 179–188. MR [2083038](#)
- [Bou04] Karim Boulabiar, *Representation theorems for d -multiplications on Archimedean unital f -rings*, Comm. Algebra **32** (2004), no. 10, 3955–3967. MR [2097440](#)
- [BP56] Garrett Birkhoff and R. S. Pierce, *Lattice-ordered rings*, An. Acad. Brasil. Ci. **28** (1956), 41–69. MR [80099](#)

- [BP05] Gerard Buskes and Robert Page, *A positive note on a counterexample by Arens*, Quaest. Math. **28** (2005), no. 1, 117–121. MR [2136186](#)
- [Buc61] R. C. Buck, *Multiplication operators*, Pacific J. Math. **11** (1961), 95–103. MR [123184](#)
- [BW17] G. J. H. M. Buskes and A. W. Wickstead, *Tensor products of f -algebras*, Mediterr. J. Math. **14** (2017), no. 2, 10. MR [3619425](#)
- [CD71] P. F. Conrad and J. E. Diem, *The ring of polar preserving endomorphisms of an abelian lattice-ordered group*, Illinois J. Math. **15** (1971), 222–240. MR [285462](#)
- [CM11] Raúl E. Curto and Martin Mathieu (eds.), *Elementary operators and their applications*, Operator Theory: Advances and Applications, vol. 212, Birkhäuser/Springer Basel AG, Basel, 2011, Papers from the 3rd International Workshop held at Queen’s University Belfast, Belfast, April 14–17, 2009. MR [2789128](#)
- [Con74] Paul Conrad, *The additive group of an f -ring*, Canadian J. Math. **26** (1974), 1157–1168. MR [354487](#)
- [Dav96] Kenneth R. Davidson, *C^* -algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR [1402012](#)
- [DdJ23] Chun Ding and Marcel de Jeu, *Direct limits in categories of normed vector lattices and Banach lattices*, Positivity **27** (2023), no. 3, Paper No. 39, 32. MR [4591541](#)
- [Ded97] R. Dedekind, *On the decomposition of numbers by means of their greatest common divisors.*, Braunsch. Festschr. 1-40 (1897)., 1897.
- [dHT23] David de Hevia and Pedro Tradacete, *Free complex Banach lattices*, J. Funct. Anal. **284** (2023), no. 10, 26. MR [4552375](#)
- [dHT25] David de Hevia and Pedro Tradacete, *Complemented subspaces of Banach lattices*, Banach J. Math. Anal. **19** (2025), no. 4, 33. MR [4940175](#)
- [dJ21] Marcel de Jeu, *Free vector lattices and free vector lattice algebras*, Positivity and its applications, Trends Math., Birkhäuser/Springer, Cham, 2021, pp. 103–139. MR [4383622](#)
- [dJML] Marcel de Jeu and David Muñoz-Lahoz, *Representation theory of vector lattice algebras*, Manuscript in preparation.

- [DMCRARZ21] Sheldon Dantas, Gonzalo Martínez-Cervantes, José David Rodríguez Abellán, and Abraham Rueda Zoca, *Octahedral norms in free Banach lattices*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **115** (2021), no. 1, 20. MR [4165729](#)
- [DMCRARZ22] Sheldon Dantas, Gonzalo Martínez-Cervantes, José David Rodríguez Abellán, and Abraham Rueda Zoca, *Norm-attaining lattice homomorphisms*, Rev. Mat. Iberoam. **38** (2022), no. 3, 981–1002. MR [4413761](#)
- [DP81] B De Pagter, *f-algebras and orthomorphisms*, 1981, PhD Dissertation (Universiteit Leiden).
- [dP84] B. de Pagter, *The space of extended orthomorphisms in a Riesz space*, Pacific J. Math. **112** (1984), no. 1, 193–210. MR [739146](#)
- [dPW15] Ben de Pagter and Anthony W. Wickstead, *Free and projective Banach lattices*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 1, 105–143. MR [3304578](#)
- [EG24] Eduard Emelyanov and Svetlana Gorokhova, *Free uniformly complete vector lattices*, Positivity **28** (2024), no. 4, Paper No. 48, 9. MR [4769232](#)
- [EH18] Karin Erdmann and Thorsten Holm, *Algebras and representation theory*, Springer Undergraduate Mathematics Series, Springer, Cham, 2018. MR [3837168](#)
- [Fuc63] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford; Addison-Wesley Publishing Co., Inc., Reading, Mass.-Palo Alto, Calif.-London, 1963. MR [171864](#)
- [GdR72] Alain Goulet de Rugy, *La structure idéale des M-espaces*, J. Math. Pures Appl. (9) **51** (1972), 331–373. MR [385512](#)
- [GK03] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, vol. 159, 2003, Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday, pp. 121–141. MR [2030906](#)
- [GSdHT24] Enrique García-Sánchez, David de Hevia, and Pedro Tradacete, *Free objects in Banach space theory*, Cutting-edge mathematics, RSME Springer Ser., vol. 13, Springer, Cham, 2024, pp. 100–124. MR [4823784](#)
- [GST24] E. García-Sánchez and P. Tradacete, *Free dual spaces and free Banach lattices*, J. Math. Anal. Appl. **532** (2024), no. 2, 22. MR [4677159](#)
- [Hay77] Richard Haydon, *Injective Banach lattices*, Math. Z. **156** (1977), no. 1, 19–47. MR [473776](#)

- [HdP82] C. B. Huijsmans and B. de Pagter, *Ideal theory in f -algebras*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 225–245. MR [637036](#)
- [HdP84a] C. B. Huijsmans and B. de Pagter, *The order bidual of lattice ordered algebras*, J. Funct. Anal. **59** (1984), no. 1, 41–64. MR [763776](#)
- [HdP84b] C. B. Huijsmans and B. de Pagter, *Subalgebras and Riesz subspaces of an f -algebra*, Proc. London Math. Soc. (3) **48** (1984), no. 1, 161–174. MR [721777](#)
- [Hen97] Melvin Henriksen, *A survey of f -rings and some of their generalizations*, Ordered algebraic structures (Curaçao, 1995), Kluwer Acad. Publ., Dordrecht, 1997, pp. 1–26. MR [1445106](#)
- [HI62] Melvin Henriksen and J. R. Isbell, *Lattice-ordered rings and function rings*, Pacific J. Math. **12** (1962), 533–565. MR [153709](#)
- [HJ62] M. Henriksen and D. G. Johnson, *On the structure of a class of archimedean lattice-ordered algebras*, Fund. Math. **50** (1961/62), 73–94. MR [133698](#)
- [Hui76] C. B. Huijsmans, *Riesz spaces for which every ideal is a projection band*, Indag. Math. **38** (1976), no. 1, 30–35, Nederl. Akad. Wetensch. Proc. Ser. A **79**. MR [404092](#)
- [Hui85] C. B. Huijsmans, *An inequality in complex Riesz algebras*, Studia Sci. Math. Hungar. **20** (1985), no. 1-4, 29–32. MR [886000](#)
- [Hui88] C. B. Huijsmans, *Elements with unit spectrum in a Banach lattice algebra*, Nederl. Akad. Wetensch. Indag. Math. **50** (1988), no. 1, 43–51. MR [934473](#)
- [Hui89] C. B. Huijsmans, *The order bidual of lattice ordered algebras. II*, J. Operator Theory **22** (1989), no. 2, 277–290. MR [1043728](#)
- [Hui95] C. B. Huijsmans, *A characterization of complex lattice homomorphisms on Banach lattice algebras*, Quaestiones Math. **18** (1995), no. 1-3, 131–140, First International Conference in Abstract Algebra (Kruger Park, 1993). MR [1340473](#)
- [IT20] Dijana Ilišević and Aleksej Turnšek, *On Wigner’s theorem in smooth normed spaces*, Aequationes Math. **94** (2020), no. 6, 1257–1267. MR [4171843](#)
- [Jab20] Jamel Jaber, *The Fremlin projective tensor product of Banach lattice algebras*, J. Math. Anal. Appl. **488** (2020), no. 2, 9. MR [4081548](#)

-
- [JSLT⁺22] Héctor Jardón-Sánchez, Niels Jakob Laustsen, Mitchell A. Taylor, Pedro Tradacete, and Vladimir G. Troitsky, *Free Banach lattices under convexity conditions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (2022), no. 1, 25. MR [4326041](#)
 - [Kak41a] Shizuo Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. (2) **42** (1941), 523–537. MR [4095](#)
 - [Kak41b] Shizuo Kakutani, *Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.)*, Ann. of Math. (2) **42** (1941), 994–1024. MR [5778](#)
 - [Kei95] Klaus Keimel, *Some trends in lattice-ordered groups and rings*, Lattice theory and its applications (Darmstadt, 1991), Res. Exp. Math., vol. 23, Heldermann, Lemgo, 1995, pp. 131–161. MR [1366870](#)
 - [Kri74] J. L. Krivine, *Théorèmes de factorisation dans les espaces réticulés*, Séminaire Maurey-Schwartz 1973–1974: Espaces L^p , applications radonifiantes et géométrie des espaces de Banach, École Polytech., Paris, 1974, pp. Exp. Nos. 22 et 23, 22. MR [440334](#)
 - [LSS15] F. Lucas, D. Schaub, and M. Spivakovsky, *On the Pierce-Birkhoff conjecture*, J. Algebra **435** (2015), 124–158. MR [3343214](#)
 - [LT79] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 97, Springer-Verlag, Berlin-New York, 1979, Function spaces. MR [540367](#)
 - [LT20] Niels Jakob Laustsen and Vladimir G. Troitsky, *Vector lattices admitting a positively homogeneous continuous function calculus*, Q. J. Math. **71** (2020), no. 1, 281–294. MR [4077195](#)
 - [LZ71] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1971. MR [511676](#)
 - [Mad11] James J. Madden, *Henriksen and Isbell on f -rings*, Topology Appl. **158** (2011), no. 14, 1768–1773. MR [2823688](#)
 - [Mah07] Louis Mahé, *On the Pierce-Birkhoff conjecture in three variables*, J. Pure Appl. Algebra **211** (2007), no. 2, 459–470. MR [2340463](#)

- [Mar80] Laura Martignon, *Banach f -algebras and Banach lattice algebras with unit*, Bol. Soc. Brasil. Mat. **11** (1980), no. 1, 11–17. MR [607012](#)
- [Mat92] M. Mathieu (ed.), *Proceedings of the International Workshop on Elementary Operators and Applications, Blaubeuren, 1991*, World Scientific, Singapore, 1992.
- [ML26a] David Muñoz-Lahoz, *f -algebra products on AL and AM -spaces*, Preprint, arXiv:2507.08435. To appear in Math. Z., 2026.
- [ML26b] David Muñoz-Lahoz, *Lattice-ordered algebras admitting a polynomial growth continuous function calculus*, Preprint, arXiv:2604.20294, 2026.
- [ML26c] David Muñoz-Lahoz, *Wickstead’s conjecture on positive projections and non-representable banach lattice algebras*, Preprint, arXiv:2604.14697, 2026.
- [MLT26] David Muñoz-Lahoz and Pedro Tradacete, *Free banach f -algebras*, Preprint, arXiv:2511.13299, 2026.
- [MN91] Peter Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991. MR [1128093](#)
- [MnL25] David Muñoz Lahoz, *Band projections and order idempotents in Banach lattice algebras*, J. Math. Anal. Appl. **543** (2025), no. 2, part 3, Paper No. 129045, 27. MR [4824648](#)
- [MnLT24] David Muñoz Lahoz and Pedro Tradacete, *Band projections in spaces of regular operators*, Trans. Amer. Math. Soc. **377** (2024), no. 7, 5197–5218. MR [4778072](#)
- [MnLT25] David Muñoz Lahoz and Pedro Tradacete, *Banach lattice AM -algebras*, Proc. Amer. Math. Soc. **153** (2025), no. 6, 2565–2577. MR [4892628](#)
- [MT20] Martin Mathieu and Pedro Tradacete, *Strictly singular multiplication operators on $\mathcal{L}(X)$* , Israel J. Math. **236** (2020), no. 2, 685–709. MR [4093900](#)
- [Muj91] Jorge Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 867–887. MR [1000146](#)
- [OTTT24] T. Oikhberg, M. A. Taylor, P. Tradacete, and V. G. Troitsky, *Free Banach lattices*, J. Eur. Math. Soc. (2024).

- [Pal94] Theodore W. Palmer, *Banach algebras and the general theory of $\ast$ -algebras. Vol. I*, Encyclopedia of Mathematics and its Applications, vol. 49, Cambridge University Press, Cambridge, 1994, Algebras and Banach algebras. MR [1270014](#)
- [Ren92] H. Render, *Lattice structures of ordered Banach algebras*, Illinois J. Math. **36** (1992), no. 2, 238–250. MR [1156625](#)
- [Rie40] Frederic Riesz, *Sur quelques notions fondamentales dans la théorie générale des opérations linéaires*, Ann. Math. (2) **41** (1940), 174–206 (French).
- [RS64] K. A. Ross and A. H. Stone, *Products of separable spaces*, Amer. Math. Monthly **71** (1964), 398–403. MR [164314](#)
- [Sch74] Helmut H. Schaefer, *Banach lattices and positive operators*, Die Grundlehren der mathematischen Wissenschaften, Band 215, Springer-Verlag, New York-Heidelberg, 1974. MR [423039](#)
- [Sch81] Egon Scheffold, *FF-Banachverbandsalgebren*, Math. Z. **177** (1981), no. 2, 193–205. MR [612873](#)
- [Sch84] Egon Scheffold, *Banachverbandsalgebren mit einer natürlichen Wedderburn-Zerlegung*, Math. Z. **185** (1984), no. 4, 521–531. MR [733772](#)
- [Sch91] Egon Scheffold, *Der Bidual von F -Banachverbandsalgebren*, Acta Sci. Math. (Szeged) **55** (1991), no. 1-2, 167–179. MR [1124955](#)
- [Sch93] Egon Scheffold, *über den ordnungstetigen Bidual von FF-Banachverbandsalgebren*, Arch. Math. (Basel) **60** (1993), no. 5, 473–477. MR [1213518](#)
- [She51] S. Sherman, *Order in operator algebras*, Amer. J. Math. **73** (1951), 227–232. MR [42065](#)
- [SWA78] Helmut H. Schaefer, Manfred Wolff, and Wolfgang Arendt, *On lattice isomorphisms with positive real spectrum and groups of positive operators*, Math. Z. **164** (1978), no. 2, 115–123. MR [517148](#)
- [Syn80] J. Synnatzschke, *Über einige verbandstheoretische Eigenschaften der Multiplikation von Operatoren in Vektorverbänden*, Math. Nachr. **95** (1980), 273–292. MR [592901](#)
- [Tro19] V. G. Troitsky, *Simple constructions of $\text{FBL}(A)$ and $\text{FBL}[E]$* , Positivity **23** (2019), no. 5, 1173–1178. MR [4011244](#)
- [Val67] Klaus Vala, *Sur les éléments compacts d’une algèbre normée*, Ann. Acad. Sci. Fenn. Ser. A I **407** (1967), 8. MR [222642](#)

- [Wag10] Sven Wagner, *On the Pierce-Birkhoff conjecture for smooth affine surfaces over real closed fields*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), 221–242. MR [2675729](#)
- [Wic77] A. W. Wickstead, *Representation and duality of multiplication operators on Archimedean Riesz spaces*, Compositio Math. **35** (1977), no. 3, 225–238. MR [454728](#)
- [Wic80] A. W. Wickstead, *Extensions of orthomorphisms*, J. Austral. Math. Soc. Ser. A **29** (1980), no. 1, 87–98. MR [566279](#)
- [Wic87] A. W. Wickstead, *The injective hull of an Archimedean f -algebra*, Compositio Math. **62** (1987), no. 3, 329–342. MR [901395](#)
- [Wic88] A. W. Wickstead, *Banach lattices with trivial centre*, Proc. Roy. Irish Acad. Sect. A **88** (1988), no. 1, 71–83. MR [974286](#)
- [Wic15] A. W. Wickstead, *Norms of basic elementary operators on algebras of regular operators*, Proc. Amer. Math. Soc. **143** (2015), no. 12, 5275–5280. MR [3411145](#)
- [Wic17a] A. W. Wickstead, *Banach lattice algebras: some questions, but very few answers*, Positivity **21** (2017), no. 2, 803–815. MR [3656022](#)
- [Wic17b] A. W. Wickstead, *Ordered Banach algebras and multi-norms: some open problems*, Positivity **21** (2017), no. 2, 817–823. MR [3656023](#)
- [Wic17c] A. W. Wickstead, *Two dimensional unital Riesz algebras, their representations and norms*, Positivity **21** (2017), no. 2, 787–801. MR [3656021](#)
- [Wil71] Wilbert Wils, *The ideal center of partially ordered vector spaces*, Acta Math. **127** (1971), 41–77. MR [463881](#)
- [Yud39] A. J. Yudin, *Solutions de deux problèmes de la théorie des espaces semi-ordonnés*, Doklady Akademii Nauk SSSR **23** (1939), 418–422 (French).
- [Zaa75] A. C. Zaanen, *Examples of orthomorphisms*, J. Approximation Theory **13** (1975), 192–204. MR [355527](#)
- [Zaa83] A. C. Zaanen, *Riesz spaces. II*, North-Holland Mathematical Library, vol. 30, North-Holland Publishing Co., Amsterdam, 1983. MR [704021](#)
- [Zaa97] Adriaan C. Zaanen, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, 1997. MR [1631533](#)